

Limit of Universality of Entropy-Area Law for Multi-Horizon Spacetimes ¹

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Abstract

It may be a common understanding at present that, once event horizons are in thermal equilibrium, the entropy-area law holds inevitably. However, no rigorous verification is given to such a very strong universality of the law in multi-horizon spacetimes. In this article, based on thermodynamically consistent and rigorous discussion, we investigate thermodynamics of Schwarzschild-de Sitter spacetime in which the temperatures of two horizons are different. We recognize that *three* independent state variables exist in thermodynamics of the horizons. One of the three variables represents the effect of “external gravity” acting on one horizon due to another one. Then we find that thermodynamic formalism with three independent variables suggests the breakdown of entropy-area law, and clarifies the necessary and sufficient condition for the entropy-area law. As a by-product, the special role of cosmological constant in thermodynamics of horizons is also revealed. Finally we propose two discussions; one of them is on the quantum statistics of underlying quantum gravity, and another is on the Schwarzschild-de Sitter black hole evaporation from the point of view of non-equilibrium thermodynamics.

1 Introduction

In searching for quantum theory of gravity, many plausible ideas have been proposed. However, non of them is completed. Each of the present incomplete theory includes conceptual and/or technical difficult problems. Now, it may be useful and meaningful to refine reliable and rigorous basis of the search for complete quantum gravity. The black hole thermodynamics, especially the so-called entropy-area law, seems to be one of the reliable and rigorous basis. This article inspects the range of validity of the entropy-area law, and reveal a rather unexpected limit of the law for multi-horizon spacetimes. Discussions in this article are based on two papers [1, 2].

The entropy-area law, which is regarded as an equation of state for an event horizon, claims the equilibrium entropy of event horizon is equal to one-quarter of its spatial area in Planck units [3, 4, 5, 6]. This law is already verified for spacetimes possessing a *single* event horizon [7, 8, 9, 1]. Then we may naively expect that the entropy-area law holds also for multi-horizon spacetimes, once every horizon is individually in thermal equilibrium. This expectation is equivalent to consider that the thermal equilibrium of each horizon is the necessary and sufficient condition to ensure the entropy-area law for each horizon. However this expectation has not been rigorously verified in multi-horizon spacetimes. (Comments on existing researches on Schwarzschild-de Sitter spacetime will be given later in this section.) At present, there remains the possibility that the thermal equilibrium may be simply the necessary condition of entropy-area law. If we find an example that some event horizon does not satisfy the entropy-area law even when it is in thermal equilibrium, then we recognize the thermal equilibrium as simply the necessary condition of the entropy-area law.

We can consider a situation in which the entropy-area law may break down in multi-horizon spacetime ³. To explain it, it is necessary to *distinguish* thermodynamic state of each horizon and that of the

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³ All discussions in this article are based on the ordinary general relativity. The other modified theories of gravity are not considered. Even if there is a breakdown of entropy-area law due to exotic fields of modified theory, such a breakdown in modified theory is out of the scope of this article.

total system (multi-horizon spacetime) composed of several horizons. Even when every horizon in a multi-horizon spacetime is in an equilibrium state *individually*, the total system composed of several horizons is never in any equilibrium state if the equilibrium state of one horizon is different from that of the other horizon. For example, if the temperatures of horizons in a multi-horizon spacetime are different from each other, then a net energy flow arises from a high temperature horizon to a low temperature one. Such multi-horizon spacetime can not be understood to be in any equilibrium state, since, exactly speaking, no energy flow arises in thermal equilibrium states. The thermal equilibrium of total system (multi-horizon spacetime) is realized if and only if the temperatures of all constituent horizons are equal. Therefore, when the temperatures of horizons are not equal, the multi-horizon spacetime should be understood as it is in a *non-equilibrium* state. Here it should be noticed that, generally in non-equilibrium physics, once the system under consideration becomes non-equilibrium, the equation of state for non-equilibrium case takes different form in comparison with that for equilibrium case. Especially the non-equilibrium entropy deviates from the equilibrium entropy (when a non-equilibrium entropy is well defined). Indeed, although a quite general formulation of non-equilibrium thermodynamics remains unknown at present, the difference of non-equilibrium entropy from equilibrium one is already revealed for some restricted class of non-equilibrium systems [10, 11]. Hence, for multi-horizon spacetimes composed of horizons of different temperatures, it seems to be reasonable to expect the breakdown of entropy-area law. However, since a “non-equilibrium thermodynamics” applicable to multi-horizon spacetime has not been constructed at present, we need to make use of “equilibrium thermodynamics” to investigate thermodynamic properties of multi-horizon spacetimes.

Motivated by the above consideration, this article treats Schwarzschild-de Sitter (SdS) spacetime as the representative of multi-horizon spacetimes [2]. We construct *two thermal equilibrium systems* in SdS spacetime; one of them is for black hole event horizon (BEH) and another is for cosmological event horizon (CEH). Note that, since the temperature of BEH is always higher than that of CEH in SdS spacetime [12] (see Sec.5), we need a good way to obtain thermal equilibrium systems of BEH and CEH. As will be explained in detail in Sec.5, we will adopt the same way of constructing *two* thermal equilibrium systems as Gibbons and Hawking have used in calculating the Hawking temperatures of BEH and CEH [12]; it is to introduce, between BEH and CEH, a thin wall which reflects perfectly the Hawking radiation coming from the horizons. The region enclosed by the wall and BEH (CEH) settles down to a thermal equilibrium state, and we obtain two thermal equilibrium systems separated by the perfectly reflecting wall. Then we will examine the entropy-area law for the two thermal equilibrium systems *individually*. (Although we are motivated by a non-equilibrium thermodynamic consideration in previous paragraph, the whole analysis in this article is based on equilibrium thermodynamics and we discuss the *two* equilibrium thermodynamics for BEH and CEH individually.) As will be explained in Sec.5, our two thermal systems are treated in the canonical ensemble to obtain the free energies of BEH and CEH. Hence we will make use of the Euclidean action method which is regarded as one technique to obtain the partition function of canonical ensemble of quantum gravity [13] (see Sec.2 or Appendix A of papers [1, 2]). Then we will find that the free energies are functions of three independent state variables. The existence of three independent state variables for SdS spacetime was not recognized in existing works on multi-horizon spacetimes [12, 14, 15, 16, 17, 18, 19, 20]. But in this article, thermodynamically rigorous analysis with three independent state variables will suggest a reasonable evidence of breakdown of entropy-area law for CEH. The validity of the law for BEH will not be judged, but we will clarify the key issue for BEH’s entropy.

These results imply that the thermal equilibrium of each horizon may not be the necessary and sufficient condition but simply the necessary condition of entropy-area law. The necessary and sufficient condition of the law may be implied via some existing works as follows: Let us note that some proposals for thermodynamics of BEH and CEH in SdS spacetime are already given to a case with some special matter fields and for an extreme case with magnetic/electric charge [14, 15]. These examples are artificial to vanish the temperature difference of BEH and CEH, and show that the entropy-area law holds for SdS spacetime if the temperatures of BEH and CEH are equal. However, in this article, we consider a more general case which is not extremal and does not include artificial matter fields. In all analyses in this article, the temperatures of BEH and CEH remain different and the discussions in those examples [14, 15] can not be applied. If we find the breakdown of entropy-area law for the case that horizons have different

temperatures, then it is suggested that *the necessary and sufficient condition of entropy-area law is the thermal equilibrium of the total system composed of several horizons in which the net energy flow among horizons disappears.*

Here we should make comments on the case that horizons have different temperatures. The construction of SdS thermodynamics with leaving horizon temperatures different has already been tried in some existing works [12, 16, 17, 18, 19, 20]. Some of those works [12, 16, 17, 18, 19] assume some geometrical conserved quantities to be state variables of SdS spacetime, and derive the so-called mass formula which is simply a geometrical relation and looks similar to the first law of black hole thermodynamics. Remaining of those works [20] discusses the entropy of SdS spacetime with assuming a naive formula for the horizon entropy which uses the “effective surface gravity” obtained via the so-called tunneling method. However, the *thermodynamic consistency* has not been confirmed in all of those works. Here, “thermodynamic consistency” means that the state variables satisfy not only the four laws of thermodynamics but also the appropriate differential relations; for example, the differential of free energy F_o with respect to temperature T_o is equivalent to the minus of entropy, $S_o \equiv -\partial F_o / \partial T_o$ ⁴. It is obvious that thermodynamic entropy should be given in the theory satisfying thermodynamic consistency. In other words, if one asserts some theoretical framework to be a “thermodynamics”, that framework must satisfy the thermodynamic consistency. Therefore, exactly speaking, it remains unclear whether those existing works [12, 16, 17, 18, 19, 20] are appropriate as “thermodynamics” ⁵. On the other hand, it seems to be preferable that the number of assumptions for thermodynamic formulations of BEH and CEH is as small as possible. In order to introduce the minimal set of assumptions which preserves thermodynamic consistency, we will refer to Schwarzschild canonical ensemble [7] for BEH. Also we will refer to de Sitter canonical ensemble for CEH. Our conceptual essence is the thermodynamic consistency which gives thermodynamically rigorous formulation to SdS spacetime.

Here let us make a comment that de Sitter canonical ensemble has not been formulated, while its micro-canonical ensemble has been established. Therefore, this article includes the formulation of de Sitter canonical ensemble [1] before proceeding to SdS canonical ensemble.

On the other hand, some existing works [18], motivated by the so-called dS/CFT correspondence conjecture [21], focus their attention on the future and past null infinities in SdS spacetime (see Fig.3 shown in Sec.5, I^\pm is the null infinities). Those infinities may be appropriate to discuss some geometrical quantities. However, as implied by the causal structure of SdS spacetime, the future null infinity seems to be inappropriate to discuss thermodynamic properties of BEH and CEH, because any observer near future null infinity (not near the future temporal infinity i^+) can not “access” BEH ⁶.

Hence, contrary to the existing works, the analysis in this article is based on the following two points:

- As will be explained precisely in Sec.5, we focus our attention on the region enclosed by BEH and CEH (not on null infinity) in SdS spacetime as the object of thermodynamic interests.
- We have a high regard to the “thermodynamic consistency” preserved by the minimal set of assumptions without referring to some geometrical conserved quantities and dS/CFT correspondence.

Then, as the result of these two points, the evidence of breakdown of entropy-area law will be obtained.

Here let us emphasize that, in the following sections, we will exhibit explicitly the assumptions on which our discussion is based. We think readers can judge the approval or disapproval to every part of our assumptions and analyses. Therefore, even if some part of our discussion and analysis is not acceptable for some reader, we hope this article can propose one possible issue about the universality of entropy-area law.

⁴ There are many other similar differential relations in thermodynamics. Those relations are the ones required in the “thermodynamic consistency”, and necessary to understand thermodynamic properties of the system under consideration; e.g. phase transition, thermal and mechanical stabilities, and equations of states.

⁵ Some of those existing works [12, 16, 17, 18, 19] preserve/assume the entropy-area law without confirming thermodynamic consistency. Hence, if the breakdown of entropy-area law is concluded via the thermodynamic consistency, those existing works can not be regarded as “thermodynamic” theory.

⁶ The observer going towards the future temporal infinity i^+ in Fig.3 can “access” BEH, since the BEH becomes a boundary of the causally connected region of that observer (the region I in Fig.3).

This article is organized as follows: In Sec.2, we clarify the conceptual foundation of our discussion and summarize the important tools of our analysis. Sec.3 reviews the Schwarzschild thermodynamics formulated by York [7], which is the first thermodynamically rigorous formulation of black hole thermodynamics. Our discussion is based on the York’s formulation. In Sec.4 we formulate de Sitter thermodynamics in the canonical ensemble with referring to York’s Schwarzschild thermodynamics. The special role of cosmological constant in horizon thermodynamics is also revealed in that section. Sec.5 is devoted to the evidence of the breakdown of entropy-area law for a multi-horizon spacetime. Sec.6 is for the conclusion.

Throughout this article we use the Planck units, $c = G = \hbar = k_B = 1$.

2 Preliminary: Canonical Ensemble and Euclidean Action

As mentioned in fourth paragraph of Sec.1, we use the Euclidean action method which is a technique to obtain the partition function of canonical ensemble of quantum gravity [13]. For the first, this section summarizes an important meaning of partition function which forms the conceptual basis of this article. Then, a review of Euclidean action method and a summary of useful differential formulas follow.

2.1 Important Meaning of Partition Function

For the aim of this article that is the inspection of entropy-area law in the framework of black hole thermodynamics, it is important to recognize clearly the relation between “thermodynamics” and “statistical mechanics”. In statistical mechanics, the partition function can not be expressed as a “function of state variables” unless the appropriate state variables, on which the partition function depends, are specified *a priori* [22, 23]. To understand this, consider for example an ordinary gas in a spherical container of radius R , in which the number of constituent particles is N , the mass of one particle is m and the average speed of particles is v . The ordinary statistical mechanics, without the help of thermodynamics, yields the partition function $Z_{\text{gas}} = Z_{\text{gas}}(R, N, m, v)$ as simply a function of “parameters”, R , N , m and v . *Statistical mechanics, solely, can not determine what combinations of those parameters behave as state variables. To determine it, the first law of thermodynamics is necessary* [22, 23]. (Note that the notion of *heat* in the first law is established by purely the argument in thermodynamics, not in statistical mechanics.) Comparing the differential of partition function with the first law results in the identification of partition function with the free energy divided by temperature. Then, since the free energy of ordinary gases is a function of the temperature and volume due to the “thermodynamic” argument, the partition function $Z_{\text{gas}}(R, N, m, v)$ should be rearranged to be a function of temperature and volume $Z_{\text{gas}}(V, T)$, where $V = (4\pi/3)R^3$ and $T = mv^2$ for ideal gases due to the law of equipartition of energy⁷. (The dependence on N is, for example, $Z_{\text{gas}} \propto N$ for ideal gases.)

The reason why the temperature and volume are regarded as the state variables of the gas is that they are consistent with the four laws of “thermodynamics” and have the appropriate properties as state variable. The appropriate properties are that the state variables are macroscopically measurable, the state variables are classified into two categories, *intensive* variables and *extensive* variables, and the extensive variables are additive. Those properties of state variables are specified by purely the argument in thermodynamics, not in statistical mechanics. Therefore, from the above, it is recognized that statistical mechanics can not yield the partition function as a “function of appropriate state variables” without the help of thermodynamics which specifies the appropriate state variables for the partition function.

Turn our discussion to the Euclidean action method for curved spacetimes. Since the Euclidean action method is the technique to obtain the “partition function” of the spacetime under consideration (see next subsection), it is necessary to specify the state variables before calculating the Euclidean action. In this article, we formulate the canonical ensemble for de Sitter thermodynamics in Sec.4 with referring to Schwarzschild canonical ensembles [7] summarized in Sec.3, and then introduce the minimal set of assumptions for SdS thermodynamics in Sec.5, which specify the appropriate state variables for the partition function. The special role of cosmological constant [1, 2, 19] is also clarified in those discussion.

⁷ When the number of particles N changes by, for example, a chemical reaction and an exchange of particles with environment, N is also the state variable on which the free energy depends.

2.2 Euclidean Action Method

The Euclidean action method for systems including gravity is originally introduced by Gibbons and Hawking [13] in an analogy with the thermal field theory of matter fields in flat spacetime [24]. For the first, we summarize thermal fields in flat spacetime, and then review its generalization by Gibbons and Hawking.

2.2.1 Thermal fields in flat spacetime

Thermal field theory is the statistical mechanics of quantum fields in thermal equilibrium [24]. The partition function for the canonical ensemble of a field ϕ in Minkowski spacetime is defined by the path integral,

$$Z_{\text{flat}} := \int \mathcal{D}\phi e^{I_E[\phi]}, \quad (2.1)$$

where $\mathcal{D}\phi$ is a normalized measure of path integral and $I_E[\phi]$ is the Euclidean action of ϕ defined by

$$I_E[\phi] := i \times \text{Lorentzian action with replacing } t \text{ by } -i\tau, \quad (2.2)$$

where the Lorentzian metric signature is $(-+++)$, the time coordinate t in the Minkowski spacetime is of ordinary Cartesian coordinates (the time-time component of metric is -1), and the replacement of real time t by imaginary time τ is called the *Wick rotation*. By the Wick rotation $t \rightarrow -i\tau$, the metric in evaluating $I_E[\phi]$ becomes that of flat Euclidean space with signature $(++++)$. The “direction” of Wick rotation on complexified time plane is “clockwise” $t \rightarrow -i\tau$ (not “counterclockwise” $t \rightarrow +i\tau$) in order to make Z_{flat} correspond to the partition function of (grand-)canonical ensemble in quantum statistics [24]. In the path integral in Eq.(2.1), an appropriate boundary condition is also given to ϕ in order to realize a thermal equilibrium state. At least, because thermal equilibrium state is static, a periodic boundary condition in the imaginary time direction is required,

$$\phi(\tau) = \phi(\tau + \beta), \quad (2.3)$$

where β is the imaginary time period⁸. With this condition, it has already been known [24] that Z_{flat} corresponds to the partition function of canonical ensemble of equilibrium temperature T_{flat} defined by

$$T_{\text{flat}} := \frac{1}{\beta}. \quad (2.4)$$

Z_{flat} describes thermal equilibrium state of ϕ of equilibrium temperature T_{flat} in Minkowski spacetime, and the free energy F_{flat} of the equilibrium state is obtained,

$$F_{\text{flat}} = -T_{\text{flat}} \ln Z_{\text{flat}}. \quad (2.5)$$

2.2.2 Curved spacetime and thermal fields on it

In curved spacetime, we consider a thermal equilibrium state of the combined system of spacetime and matter field. For the canonical ensemble of our combined system, it is usually assumed that the partition function Z is obtained by replacing flat metric in Eq.(2.1) with curved one [13],

$$Z := \int \mathcal{D}g_E \cdot \mathcal{D}\phi e^{I_E[g_E, \phi]}, \quad (2.6)$$

where

$$I_E[g_E, \phi] := i \times I_{\text{tot}}[g, \phi] \text{ with Wick rotation } t \rightarrow -i\tau, \quad (2.7)$$

where $I_{\text{tot}}[g, \phi]$ is the Lorentzian action explained below, and g_E is the Euclidean metric of signature $(++++)$ obtained from the Lorentzian metric g by the Wick rotation $t \rightarrow -i\tau$. Note that, since the

⁸ When the periodic boundary condition in imaginary time is not required, the path integral in Eq.(2.1) describes an ordinary transition amplitude of ϕ in ordinary quantum field theory of zero temperature.

spacetime metric g is also assumed to be quantum metric, g_E appears as an integral variable in the path integral (2.6). The action is given as

$$I_{\text{tot}}[g, \phi] := I_{\text{matter}}[g, \phi] + I_L[g], \quad (2.8)$$

where $I_{\text{matter}}[g, \phi]$ is the Lorentzian matter action, and $I_L[g]$ is the Lorentzian Einstein-Hilbert action defined as

$$I_L := \frac{1}{16\pi} \int_{\mathcal{M}} dx^4 \sqrt{-\det g} (\mathcal{R} - 2\Lambda) + \frac{1}{8\pi} \int_{\partial\mathcal{M}} dx^3 \sqrt{\det h} K + I_{\text{sub}}, \quad (2.9)$$

where \mathcal{M} is the spacetime region under consideration, \mathcal{R} is the Ricci scalar, Λ is the cosmological constant, h and K in the second term are respectively the first fundamental form (induced metric) and the trace of second fundamental form (extrinsic curvature) of the boundary $\partial\mathcal{M}$, and I_{sub} is the integration constant of I_L which is sometimes called the subtraction term. The second term $\int_{\partial\mathcal{M}}$ in Eq.(2.9) is required to eliminate the second derivatives of metric from the action [25]. The third term I_{sub} does not contribute to the Einstein equation obtained by $\delta_g I_{\text{tot}} = 0$.

In order to consider equilibrium states of spacetime with matter field, the periodic boundary condition in imaginary time is required for not only ϕ but also g_E ,

$$g_{E\mu\nu}(\tau) = g_{E\mu\nu}(\tau + \beta). \quad (2.10)$$

Here the equilibrium temperature of g and ϕ is not defined simply by β^{-1} , because the spacetime is curved. Instead of the simple inverse β^{-1} , the temperature T should be defined by the proper length in the Euclidean space of g_E in the imaginary time direction,

$$T := \left[\int_0^\beta \sqrt{g_{E\tau\tau}} d\tau \right]^{-1}. \quad (2.11)$$

Since the metric component $g_{E\tau\tau}$ is a function of spacetime coordinates, the integral in Eq.(2.11) becomes a function of spatial coordinates. Therefore it is important to specify where the temperature is defined. Here note that the Euclidean action method is for the canonical ensemble. This implies the existence of a heat bath whose temperature coincides with the temperature of the system under consideration, since the system is in a thermal equilibrium with the heat bath. Therefore it is reasonable to evaluate $g_{E\tau\tau}$ in Eq.(2.11) at the contact surface of the system with the heat bath. The contact surface is the boundary of the spacetime region. Hence the temperature T should be evaluated at the spacetime boundary.

In the path integral in Eq.(2.6), the metric and matter field are not necessarily solutions of classical Einstein equation and field equations. However when the field ϕ is weak enough, the dominant contribution would come from the classical solutions, g_{cl} and ϕ_{cl} , and we can expand as

$$g_{\mu\nu} = g_{cl\mu\nu} + \delta g_{\mu\nu}, \quad \phi = \phi_{cl} + \delta\phi, \quad (2.12)$$

where δg and $\delta\phi$ describe quantum/statistical fluctuations of metric and matter. In spacetimes with event horizon, this expansion seems reasonable since the Hawking temperature is usually very low and the matter field ϕ of Hawking radiation is weak. Then the Euclidean action becomes

$$I_E[g_E, \phi] = I_E[g_{Ecl}, \phi_{cl}] + I_E[\delta g_E] + I_E[\delta\phi] + \text{higher order terms}, \quad (2.13)$$

where g_{Ecl} is the Euclidean metric obtained from g_{cl} , and the second and third terms are quadratic in fluctuations by definition of classical field equations. The partition function becomes

$$\ln Z = I_E[g_{Ecl}, \phi_{cl}] + \ln \int \mathcal{D}(\delta g_E) e^{I_E[\delta g_E]} + \ln \int \mathcal{D}(\delta\phi) e^{I_E[\delta\phi]} + \dots. \quad (2.14)$$

The leading term $I_E[g_{Ecl}, \phi_{cl}]$ includes only the classical solutions. Hence the partition function Z_{cl} of the thermal equilibrium state of background classical spacetime and matter is defined by

$$\ln Z_{cl} := I_E[g_{Ecl}, \phi_{cl}]. \quad (2.15)$$

The state variables obtained from Z_{cl} describe thermal equilibrium states of background spacetime and matter. For spacetimes with event horizon, Z_{cl} is interpreted as the partition function of the event horizon. For empty background spacetimes ($\phi_{cl} = 0$) like Schwarzschild and de Sitter spacetimes, Z_{cl} is determined by only classical metric, $\ln Z_{cl} = I_E[g_{Ecl}]$. This $I_E[g_{Ecl}]$ describes the canonical ensemble of the thermal equilibrium states of the background classical spacetime, where the thermal equilibrium is achieved by the interaction with the quantum fluctuations of metric. Then the free energy of those classical background spacetimes are determined by

$$F = -T \ln Z_{cl} = -T I_E[g_{Ecl}], \quad (2.16)$$

where T is defined by Eq.(2.11) with replacing g_E by g_{Ecl} . This T is the equilibrium temperature of the event horizon.

Finally let us make a comment: In this article, we take the standpoint that the Euclidean action method is simply one (promising) formalism of obtaining the partition function of spacetimes. At present, since we do not know a complete quantum gravity theory, the use of Euclidean action is to be understood as one assumption.

2.3 Useful Differential Relations

Let us exhibit useful differential formulas for calculations of thermodynamic state variables. The reader can skip this subsection and return here when the formulas displayed below are referred in the analysis in following sections.

2.3.1 First case

Let f be a function of α_1 , α_2 and α_3 , $f = f(\alpha_1, \alpha_2, \alpha_3)$. And consider the case that α_i ($i = 1, 2, 3$) are also functions of y_1 , y_2 and y_3 , $\alpha_i = \alpha_i(y_j)$ ($j = 1, 2, 3$). Then define $f(y_1, y_2, y_3) := f(\alpha_i(y_j))$. Let us aim to express the partial derivatives $\partial f(\alpha_1, \alpha_2, \alpha_3)/\partial \alpha_i$ by the derivatives with respect to y_j . Standard differential calculus gives, $\partial_{y_j} f = \sum_{i=1}^3 (\partial_{\alpha_i} f) (\partial_{y_j} \alpha_i)$, which is expressed in vector form as

$$\begin{pmatrix} \partial_{y_1} f \\ \partial_{y_2} f \\ \partial_{y_3} f \end{pmatrix} = P \begin{pmatrix} \partial_{\alpha_1} f \\ \partial_{\alpha_2} f \\ \partial_{\alpha_3} f \end{pmatrix}, \quad P := \begin{pmatrix} \partial_{y_1} \alpha_1 & \partial_{y_1} \alpha_2 & \partial_{y_1} \alpha_3 \\ \partial_{y_2} \alpha_1 & \partial_{y_2} \alpha_2 & \partial_{y_2} \alpha_3 \\ \partial_{y_3} \alpha_1 & \partial_{y_3} \alpha_2 & \partial_{y_3} \alpha_3 \end{pmatrix}. \quad (2.17)$$

Then, when $\det P \neq 0$, we obtain

$$\begin{pmatrix} \partial_{\alpha_1} f \\ \partial_{\alpha_2} f \\ \partial_{\alpha_3} f \end{pmatrix} = P^{-1} \begin{pmatrix} \partial_{y_1} f \\ \partial_{y_2} f \\ \partial_{y_3} f \end{pmatrix}. \quad (2.18)$$

2.3.2 Second case

Use the same definitions with previous subsection. If f has no α_3 -dependence ($f = f(\alpha_1, \alpha_2)$) and α_i has no y_3 -dependence ($\alpha_i = \alpha_i(y_1, y_2)$, $i = 1, 2$), then Eq.(2.18) reduces to

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_1} = \frac{(\partial_{y_1} f) (\partial_{y_2} \alpha_2) - (\partial_{y_2} f) (\partial_{y_1} \alpha_2)}{(\partial_{y_1} \alpha_1) (\partial_{y_2} \alpha_2) - (\partial_{y_2} \alpha_1) (\partial_{y_1} \alpha_2)}, \quad (2.19)$$

and a similar formula given by exchanging α_1 and α_2 .

2.3.3 Third case

Use the same definitions with previous subsection. If f has no α_3 -dependence ($f = f(\alpha_1, \alpha_2)$) and α_2 has no y_2 - and y_3 -dependence ($\alpha_2 = \alpha_2(y_1)$) while α_1 depends on all of y_j ($\alpha_1 = \alpha_1(y_1, y_2, y_3)$), then Eq.(2.17) reduces to

$$\begin{pmatrix} \partial_{y_1} f \\ \partial_{y_2} f \\ \partial_{y_3} f \end{pmatrix} = \begin{pmatrix} \partial_{y_1} \alpha_1 & \partial_{y_1} \alpha_2 \\ \partial_{y_2} \alpha_1 & 0 \\ \partial_{y_3} \alpha_1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{\alpha_1} f \\ \partial_{\alpha_2} f \end{pmatrix}. \quad (2.20)$$

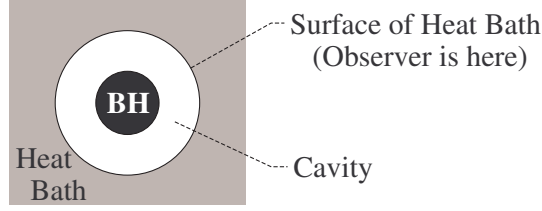


Figure 1: Schematic image of thermal equilibrium of black hole with heat bath. This is described in the canonical ensemble. State variables of black hole are defined at the surface of heat bath. With those state variables, the consistent thermodynamic formulation is realized using the Euclidean action method [7].

This gives

$$\frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_1} = \frac{\partial_{y_2} f}{\partial_{y_2} \alpha_1} = \frac{\partial_{y_3} f}{\partial_{y_3} \alpha_1} \quad (2.21)$$

$$\begin{aligned} \frac{\partial f(\alpha_1, \alpha_2)}{\partial \alpha_2} &= \frac{(\partial_{y_1} f)(\partial_{y_2} \alpha_1) - (\partial_{y_2} f)(\partial_{y_1} \alpha_1)}{(\partial_{y_1} \alpha_2)(\partial_{y_2} \alpha_1)} \\ &= \frac{(\partial_{y_1} f)(\partial_{y_3} \alpha_1) - (\partial_{y_3} f)(\partial_{y_1} \alpha_1)}{(\partial_{y_1} \alpha_2)(\partial_{y_3} \alpha_1)}. \end{aligned} \quad (2.22)$$

Furthermore, if f depends only on α_1 ($f = f(\alpha_1)$), then Eq.(2.21) reduces to

$$\frac{\partial f(\alpha_1)}{\partial \alpha_1} = \frac{\partial_{y_1} f}{\partial_{y_1} \alpha_1} = \frac{\partial_{y_2} f}{\partial_{y_2} \alpha_1} = \frac{\partial_{y_3} f}{\partial_{y_3} \alpha_1}. \quad (2.23)$$

3 York's Formulation of Schwarzschild Thermodynamics

The first thermodynamically rigorous formulation of black hole thermodynamics was given by York for Schwarzschild black hole [7]. As discussed in following sections, by referring to the York's Schwarzschild thermodynamics, we can learn the minimal set of assumptions of de Sitter and SdS thermodynamics which provide the appropriate *state variables* of partition function (Euclidean action). This section summarizes the essence of York's theory [7].

There are three key points in the canonical ensemble for Schwarzschild thermodynamics. The first one is the zeroth law which describes the existence and construction of thermal equilibrium states:

Key Point 1 (Zeroth law of black hole) *Place a black hole in a spherical cavity as shown in Fig.1 and also the observer at the surface of the heat bath. Through the Hawking radiation by black hole and the black body radiation by heat bath, the black hole interacts with the heat bath. Then, by appropriately adjusting the temperature of heat bath, the combined system of black hole and heat bath settles down to a thermal equilibrium state.*

The equilibrium state of black hole under the contact with heat bath is described in the canonical ensemble. And the equilibrium state variables of black hole are defined by the quantities measured at the surface of heat bath where the observer is. Then the “thermodynamically consistent” Schwarzschild canonical ensemble is constructed as follows.

The second key point is the difference of black hole thermodynamics from thermodynamics of ordinary laboratory systems:

Key Point 2 (Peculiar scaling law of black hole) *Extensive and intensive state variables of black hole show a peculiar scaling law: When a length size L (e.g. event horizon radius) is scaled as $L \rightarrow \lambda L$ with an arbitrary scaling rate $\lambda (> 0)$, then the extensive variables X (e.g. entropy) and intensive variables Y (e.g. temperature) are scaled as $X \rightarrow \lambda^2 X$ and $Y \rightarrow \lambda^{-1} Y$, while the thermodynamic functions Φ*

(e.g. free energy) are scaled as $\Phi \rightarrow \lambda \Phi$. This implies that, because the system size is one of the extensive variables, the thermodynamic system size of equilibrium system constructed in the key point 1 should have the areal dimension. Indeed the surface area of heat bath, $4\pi r_w^2$, behaves as the consistent extensive variable of the system size, where r_w is the radius of the surface of heat bath.

Here recall that, in thermodynamics of ordinary laboratory systems, the intensive variables remain un-scaled under the scaling of system size, the extensive variables scales as the volume, and the thermodynamic functions are the members of extensive variables. However, as noted in the key point 2, the black hole thermodynamics has the peculiar scaling law of state variables. Although the scaling law differs from that in thermodynamics of ordinary laboratory systems, the peculiar scaling law in black hole thermodynamics retains the thermodynamic consistency as noted in the next key point.

The third key point is the similarity of black hole thermodynamics with thermodynamics of ordinary laboratory systems:

Key Point 3 (Euclidean action method and thermodynamic consistency) *The free energy F_{BH} of black hole is yielded by the Euclidean action method, where the integration constant (the so-called subtraction term) of the action integral is determined with referring to flat spacetime. The action integral is evaluated in the region, $2M < r < r_w$, which is in thermal equilibrium as noted in the key point 1. Here M is the mass parameter. For the equilibrium system of Schwarzschild black hole constructed in the key point 1, the Euclidean action in Eq.(2.15) becomes*

$$I_{E(\text{BH})} = 4\pi M \left[M - 2r_w \left(1 - 2M/r_w - \sqrt{1 - 2M/r_w} \right) \right]. \quad (3.1)$$

The free energy is given by Eq.(2.16). Then, as for the ordinary thermodynamics, this free energy is expressed as a function of two independent state variables, temperature and system size;

$$F_{\text{BH}}(T_{\text{BH}}, 4\pi r_w^2), \quad (3.2)$$

where the intensive variable $T_{\text{BH}} := \left(8\pi M \sqrt{1 - 2M/r_w} \right)^{-1}$ is the Hawking temperature measured by the observer at r_w , and the factor $\sqrt{1 - 2M/r_w}$ is the so-called Tolman factor [26] which expresses the gravitational redshift affecting the Hawking radiation propagating from the black hole horizon to the observer. This Hawking temperature is obtained by Eq.(2.11). In order to let T_{BH} and $4\pi r_w^2$ be independent variables in F_{BH} , the mass parameter M and the heat bath radius r_w are regarded as two independent variables. Then the thermodynamic consistency holds as follows: The entropy S_{BH} and the “surface pressure” σ_{BH} are defined by

$$S_{\text{BH}} := -\frac{\partial F_{\text{BH}}(T_{\text{BH}}, 4\pi r_w^2)}{\partial T_{\text{BH}}} = \frac{A_{\text{BH}}}{4}, \quad \sigma_{\text{BH}} := -\frac{\partial F_{\text{BH}}(T_{\text{BH}}, 4\pi r_w^2)}{\partial (4\pi r_w^2)}, \quad (3.3)$$

where σ_{BH} has the dimension of force per unit area because the system size $4\pi r_w^2$ has the dimension of area. See Appendix B in reference [1] for a detail explanation of thermodynamic meaning of σ_{BH} . (The temperature of heat bath should be adjusted to be T_{BH} in the key point 1.) These differential relations among the free energy, entropy and surface pressure are the same with those obtained in thermodynamics of ordinary laboratory systems. Furthermore, as for the ordinary thermodynamics, the internal energy and the other thermodynamic functions are defined by the Legendre transformation of the free energy; for example the internal energy E_{BH} is

$$E_{\text{BH}}(S_{\text{BH}}, 4\pi r_w^2) := F_{\text{BH}} + T_{\text{BH}} S_{\text{BH}}. \quad (3.4)$$

The enthalpy, Gibbs energy and so on are also defined by the Legendre transformation. Then the differential relations among those thermodynamic functions and the other state variables also hold, for example $T_{\text{BH}} \equiv \partial E_{\text{BH}} / \partial S_{\text{BH}}$. Furthermore, with the state variables obtained above, we can check that the first, second and third laws of thermodynamics hold for black holes.

The above three key points hold also for the other single-horizon black hole spacetimes, and those black hole thermodynamics has already been established [7, 8, 9].

Here let us remark about the heat bath introduced in the key point 1. In York's consistent black hole thermodynamics [7], the heat bath is essential to establish the thermodynamic consistency in the canonical ensemble as explained below: Generally in thermodynamics, as noted in the key point 3, thermodynamic functions are defined as a function of two independent state variables. Especially the free energy should be expressed as a function of the temperature and the extensive state variable which represents the system size. This thermodynamic requirement is satisfied by introducing the heat bath, which gives us two independent variables; the mass parameter M and the radius of heat bath r_w . These two independent variables make it possible to define the temperature T_{BH} and the surface area $4\pi r_w^2$ as the two independent state variables in free energy $F_{\text{BH}}(T_{\text{BH}}, 4\pi r_w^2)$. Therefore the heat bath is necessary to establish manifestly the thermodynamic consistency.

Concerning the heat bath, let us make another comment here. It is possible to take the limit $r_w \rightarrow \infty$ after constructing the consistent black hole thermodynamics with the heat bath of finite r_w . Here one may think that the limit $r_w \rightarrow \infty$ corresponds to the micro-canonical ensemble, since state variables are expressed as functions of only one parameter M and the heat bath seems to disappear (run away to infinitely distant region). However it should be emphasized that, generally in statistical mechanics, the micro-canonical ensemble is not some limiting case of the canonical ensemble. Therefore the limit $r_w \rightarrow \infty$ does not mean to consider the micro-canonical ensemble (the system without heat bath), but it is just the large limit of the cavity size in the canonical ensemble. Hence the black hole thermodynamics have been established in the canonical ensemble.

4 de Sitter Thermodynamics in the Canonical Ensemble

Recall that the aim of this article is to inspect the entropy-area law for SdS spacetime [2], in which we will refer to the canonical ensembles for Schwarzschild and de Sitter spacetimes. However, the established formalism of cosmological event horizon (CEH) in de Sitter spacetime is based on the micro-canonical ensemble [15, 27]. Here note that, generally in the ordinary thermodynamics and statistical mechanics, both of the micro-canonical and canonical ensembles yield the same equation of state for any thermodynamic system. This implies the existence of de Sitter canonical ensemble. Hence, in this section, we re-formulate the thermodynamics of de Sitter CEH in the canonical ensemble [1].

4.1 Micro-Canonical Ensemble

For the first in this section, let us summarize the micro-canonical ensemble of de Sitter horizon. Thermodynamics of CEH in de Sitter spacetime can be formulated for the region bounded by the CEH solely (the region I in Fig.2), which is filled with the Hawking radiation of CEH and settles down to a thermal equilibrium state without introducing a heat bath (see Sec.4.4 for a summary of de Sitter geometry). This region can be regarded as an isolated thermodynamic system without the contact with heat bath. This implies that thermodynamics of CEH can be established in the micro-canonical ensemble. Indeed, Hawking and Ross [15] have proposed that, when the thermal equilibrium system of a horizon is isolated and has no contact with heat bath, the Euclidean action $I_E^{(\text{micro})}$ of such system can be interpreted as the number of micro-states W or the density of micro-states [27] of underlying quantum gravity, which satisfies $W = e^{I_E^{(\text{micro})}}$. Then, because $I_E^{(\text{micro})} = \pi r_{\text{ds}}^2$ is obtained (in Sec.4.4) for the isolated system of de Sitter CEH of radius r_{ds} , the entropy-area law can be obtained by the Boltzmann's relation,

$$S_{\text{ds}} := \ln W = \pi r_{\text{ds}}^2, \quad (4.1)$$

where S_{ds} is the entropy of CEH. The entropy-area law for CEH has already been verified in the micro-canonical ensemble. This relation (4.1) should be obtained also in the canonical ensemble of de Sitter CEH.

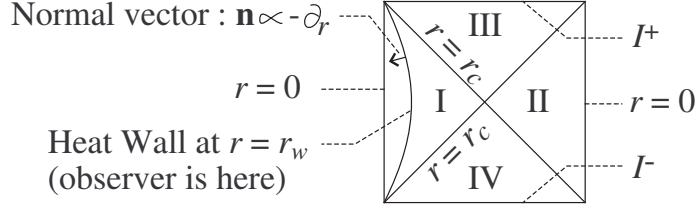


Figure 2: Penrose diagram of de Sitter spacetime for $r \geq 0$. I^\pm is the future/past null infinity. r_{ds} is the CEH radius. The heat wall of radius r_w is placed at the center of the region surrounded by CEH. Our observer is at the wall. This wall reflects perfectly the Hawking radiation, and the observer sees that the region enclosed by the wall and CEH is in a thermal equilibrium state.

4.2 Basic Assumptions of de Sitter Canonical Ensemble

Let us introduce the basic assumptions of de Sitter canonical ensemble. Those assumptions give us the way to construct thermal equilibrium state under the contact with heat bath, and specifies the appropriate state variables for the partition function (see Sec.2.1). Furthermore, to make our discussion exact logically, the use of Euclidean action method will also be listed as one assumption (see the comment at the end of Sec.2.2).

The CEH in de Sitter spacetime is a spherically symmetric null hyper-surface defined as the boundary of causal past of the observer's world line [12]. The observer detects the Hawking radiation of thermal spectrum emitted by the CEH [12]. This implies that the observer can regard the CEH as an object in thermal equilibrium. It is expected that a thermodynamically consistent canonical ensemble of CEH can be constructed in the way similar to the Schwarzschild canonical ensemble. Then, referring to the key point 1 of Schwarzschild thermodynamics, the first basic assumption of de Sitter canonical ensemble is the zeroth law:

Assumption dS – 1 (Zeroth law of CEH) *Place a spherically symmetric thin wall of radius r_w at the center of the region surrounded by the CEH as shown in Fig.2. The wall is smaller than CEH, $r_w < r_{\text{ds}}$, where r_{ds} is the CEH radius. The mass energy of this wall is negligible, and the geometry of the region $r_w < r < r_{\text{ds}}$ is of de Sitter spacetime. Let the wall reflect perfectly the Hawking radiation, and we call this perfectly reflecting wall the “heat wall” hereafter. We put our observer at the heat wall. Then, this observer sees that the region enclosed by the heat wall and CEH settles down to a thermal equilibrium state. The equilibrium state variables of CEH are defined by the quantity measured at the heat wall where the observer is.*

Since this equilibrium state of CEH has the contact with the heat wall, we expect that de Sitter thermodynamics is obtained in the canonical ensemble. However before calculating the partition function, as explained in Sec.2.1, we have to specify the appropriate state variables for the partition function, the temperature and extensive variable of system size. To do so, it is necessary to clarify the notion of extensivity and intensivity, which is the significant scaling property of state variables under the scaling of system size. The scaling behavior provides a guideline to specify the state variable of system size, and also provides the basis to adopt the temperature defined in Eq.(2.11) as an intensive state variable.

Concerning the extensivity and intensivity, note that, in general, thermodynamics seems to be a very universal formalism which can be applied to any system if it is in thermal equilibrium. This implies that the scaling behavior of state variables shown in the key point 2 of Schwarzschild thermodynamics is common to any horizon system. Then, the second assumption of de Sitter canonical ensemble is as follows:

Assumption dS – 2 (Scaling law and system size of CEH) *State variables of CEH are classified into three categories, extensive variables, intensive variables and thermodynamic functions, and those state variables satisfy the same scaling law as explained in the key point 2: When a length size L (e.g.*

CEH radius) is scaled as $L \rightarrow \lambda L$ with an arbitrary scaling rate $\lambda (> 0)$, then the extensive variables X (e.g. system size) and intensive variables Y (e.g. temperature) are scaled as $X \rightarrow \lambda^2 X$ and $Y \rightarrow \lambda^{-1} Y$, while the thermodynamic functions Φ (e.g. free energy) are scaled as $\Phi \rightarrow \lambda \Phi$. This implies that the thermodynamic system size of the equilibrium system constructed in assumption dS-1 should have the areal dimension, and the surface area of heat wall, $A_{\text{ds}} := 4\pi r_w^2$, behaves as the extensive variable of system size. (Indeed, it will be shown later that thermodynamically consistent formulation of de Sitter canonical ensemble can be established with using A_{ds} .)

The concrete functional form of free energy can not be determined by only the assumption dS-2. But, as implied by the key point 3 of Schwarzschild thermodynamics, the Euclidean action method can yield the concrete form of free energy. Hence, the third assumption of de Sitter canonical ensemble is the declaration of using the Euclidean action method:

Assumption dS – 3 (Euclidean action and State variables of CEH) *Euclidean action I_E of a thermal equilibrium state of CEH yields the partition function of canonical ensemble via Eq.(2.15), where the integration in I_E is calculated over the region enclosed by the heat wall and CEH ($r_w < r < r_{\text{ds}}$). And the free energy $F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}})$ of CEH is defined by Eq.(2.16), where T_{ds} is the temperature of CEH determined by Eq.(2.11) and A_{ds} is the area of heat wall discussed in assumption dS-2. Once F_{ds} is determined, any state variable of CEH is defined from F_{ds} in the same way with the state variable in thermodynamics of ordinary laboratory systems. For example, CEH entropy S_{ds} is defined by $S_{\text{ds}} := -\partial F_{\text{ds}}/\partial T_{\text{ds}}$.*

Let us emphasize that, exactly and logically speaking, once these assumptions are adopted, we also have to adopt supplemental working hypotheses as explained below.

4.3 Supplemental Working Hypotheses

When we require the thermodynamic consistency, the differential relations like Eq.(3.3) must be explicitly satisfied. And note that, as required in assumption dS-3, the free energy F_{ds} should be a function of two independent variables T_{ds} and A_{ds} . That is, two independent state variables must exist. On the other hand, we have only two parameters Λ and r_w in our equilibrium system constructed in assumption dS-1. Hence we have to adopt the following working hypothesis:

Working Hypothesis dS – 1 (Two independent variables) *To construct de Sitter canonical ensemble in a thermodynamically consistent way, we require that the cosmological constant Λ is an independent “working variable”. Then, we have two independent variables Λ and r_w which can ensure the free energy to be a function of two independent state variables. When we regard non-variable Λ as the physical situation, it is obtained by the “constant Λ process” in the “generalized” de Sitter thermodynamics in which Λ is regarded as a working variable to ensure the thermodynamic consistency.*

A verification of this working hypothesis will be discussed later in Sec.4.6. This working hypothesis can be combined with the assumption dS-3 to obtain a “complete free energy” as a function of two independent state variables. Here, in order to emphasize the role of Λ as a working variable, we divide the requirement of “complete free energy” into the assumption dS-3 and the working hypothesis dS-1. The assumption dS-3 and working hypothesis dS-1 may be combined to form one assumption.

Here let us comment on this working hypothesis. This article is not the first which requires the variable Λ . For example, in order to consider a thermodynamic formalism for Schwarzschild-de Sitter black hole, it has been reported that Λ has to be regarded as an variable, otherwise thermodynamic consistency is lost [18, 19]. The variable Λ seems to be a practical idea/assumption to treat CEH in thermodynamic framework.

Under the working hypothesis dS-1, the Euclidean action is to be expressed as a function of two “independent” state variables; temperature and surface area of heat wall (see assumption dS-2). In addition, the integration constant I_{sub} in Eq.(2.9) should be determined. Here recall that, as mentioned at the end of Sec.4.1, the entropy-area law should be reproduced from our Euclidean action of de Sitter canonical ensemble, since this law is the equation of state verified already in de Sitter micro-canonical ensemble [15, 27]. This requirement gives us the guiding principle to determine I_{sub} , which we summarize in the following working hypothesis:

Working Hypothesis dS – 2 (Consistency with the micro-canonical ensemble) *The entropy-area law, which is the equation of state verified already in the micro-canonical ensemble, should be reproduced in our de Sitter canonical ensemble. In other words, the integration constant I_{sub} in the action I_L should be determined so as to reproduce the entropy-area law. To do so, in calculating the Euclidean action of de Sitter spacetime, we set for the time being,*

$$I_{E\text{sub}} = \alpha_w I_E^{(\text{flat})}, \quad (4.2)$$

where $I_{E\text{sub}}$ is the integration constant in de Sitter Euclidean action, $I_E^{(\text{flat})}$ is the Euclidean action of flat spacetime ($\Lambda = 0$) whose concrete form will be shown in Sec.4.5, and α_w is a dimensionless factor. In order not to let $I_{E\text{sub}}$ affect the variational principle in obtaining Einstein equation, the factors α_w and $I_E^{(\text{flat})}$ should be expressed by only the quantity determined at the boundary of Euclidean de Sitter space (at the heat wall).⁹

Let us emphasize that Eq.(4.2) is just a working hypothesis including the unknown factor α_w . It seems that, even if one starts calculations of de Sitter Euclidean action with $I_{E\text{sub}}$ which is proportional (not to $I_E^{(\text{flat})}$ but) to some other action integral determined by only the boundary of Euclidean space, then the requirement of preserving the entropy-area law results in the same form with our $I_{E\text{sub}}$ obtained in Sec.4.5. The essence of the working hypothesis dS-2 is not Eq.(4.2) but the preservation of entropy-area law which is the equation of state verified already in the micro-canonical ensemble. This working hypothesis dS-2 is based on the statistical mechanical requirement that both the micro-canonical and canonical ensembles yield the same equation of state for any thermodynamic system [27, 22].

4.4 Euclidean Action of de Sitter canonical ensemble

4.4.1 Euclidean de Sitter space

The Lorentzian de Sitter metric in static chart is

$$ds^2 = -f_{\text{ds}}(r) dt^2 + \frac{dr^2}{f_{\text{ds}}(r)} + r^2 d\Omega^2, \quad (4.3)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on a unit two-sphere, and

$$f_{\text{ds}}(r) := 1 - H^2 r^2, \quad 3H^2 = \Lambda. \quad (4.4)$$

The Penrose diagram for $r \geq 0$ is shown in Fig.2. The static chart with $0 \leq r < H^{-1}$ covers the region I or II shown in Fig.2. The notion of CEH is observer dependent [12]. For the observer whose world line is confined in the region I, the radius r_{ds} of CEH is given by $f_{\text{ds}}(r_{\text{ds}}) = 0$;

$$r_{\text{ds}} = \frac{1}{H}. \quad (4.5)$$

A Killing vector $\xi \propto \partial_t$ becomes null at the CEH. This means that the CEH is the Killing horizon of ξ . Surface gravity κ of the Killing horizon, CEH, is defined by the relation, $\nabla_\xi \xi = \kappa \xi$ at the CEH. The value of κ depends on the normalization of ξ by definition [28]. When the Killing vector is normalized as $\xi = \partial_t$, we get

$$\kappa = H. \quad (4.6)$$

This κ is the surface gravity of CEH measured by the observer at the central world line $r = 0$, because t is the proper time of the central observer and the norm $\xi^2 = -f_{\text{ds}}(r)$ is -1 at $r = 0$.

The global chart of Lorentzian de Sitter spacetime is introduced by the coordinate transformation from (t, r, θ, φ) to (W, X, θ, φ) ,

$$W - X := e^{H(t-r^*)}, \quad W + X := -e^{-H(t+r^*)}, \quad (4.7)$$

⁹ Indeed, as will be shown in Sec.4.5, $I^{(\text{flat})}$ is expressed by only such quantity. The concrete form of α_w will be obtained in Sec.4.5.

where $dr^* := dr/f_{\text{ds}}(r)$ which means $r^* = (1/2H) \ln |(1 + Hr)/(1 - Hr)|$. This transformation yields

$$ds^2 = \frac{1}{H^2 (W^2 - X^2 - 1)^2} \left[-dW^2 + dX^2 + \frac{(W^2 - X^2 + 1)^2}{4} d\Omega^2 \right]. \quad (4.8)$$

The coordinate transformation (4.7) implies the range of coordinates, $X < W < -X$ and $X < 0$. By extending it to the range $-\infty < W < \infty$ and $-\infty < X < \infty$, the global chart covers the whole region of maximally extended de Sitter spacetime, I, II, III and IV shown in Fig.2.

Here concerning the relation of the two charts, note that the Penrose diagram shown in Fig.2 is depicted according to the global chart (4.8). Under the coordinate transformation (4.7), the direction of timelike Killing vector $\xi := \partial_t$ is future pointing in the region I, and past pointing in the region II. Therefore, throughout this section, we consider the region I in using the static chart¹⁰.

Next we proceed to the construction of Euclidean de Sitter space. As explained in Sec.2.2, we apply the Wick rotations $t \rightarrow -i\tau$ in the static chart and $W \rightarrow -iw$ in the global chart to obtain the Euclidean de Sitter space. These Wick rotations are equivalent, because the transformation (4.7), $W = e^{-Hr^*} \sinh(Ht)$, implies that the imaginary time w in global chart is defined by $w := e^{-Hr^*} \sin(H\tau)$, where τ is the imaginary time in the static chart. The Euclidean metric in the static chart is

$$ds_E^2 = f_{\text{ds}}(r) d\tau^2 + \frac{dr^2}{f_{\text{ds}}(r)} + r^2 d\Omega^2. \quad (4.9)$$

The Euclidean metric in the global chart is

$$ds_E^2 = \frac{1}{H^2 (w^2 + X^2 + 1)^2} \left[dw^2 + dX^2 + \frac{(w^2 + X^2 - 1)^2}{4} d\Omega^2 \right]. \quad (4.10)$$

About the global chart, we get from the coordinate transformation (4.7),

$$w^2 + X^2 = \frac{1 - Hr}{1 + Hr}. \quad (4.11)$$

Because of $w^2 + X^2 \geq 0$, the Euclidean de Sitter space corresponds to the region I (or II), $0 < r < r_{\text{ds}}$, in the Lorentzian de Sitter spacetime. Because the Lorentzian de Sitter spacetime is regular at $r = 0$ and $r = r_{\text{ds}}$, the Euclidean de Sitter space is also regular at those points. To examine the regularity of Euclidean space, we make use of the metric in static chart (4.9). The regularity at $r = 0$ is rather obvious, since the metric near $r = 0$ is flat and regular, $ds_E^2 \simeq d\tau^2 + dr^2 + r^2 d\Omega^2$.

To examine the regularity of Euclidean space at $r = r_{\text{ds}}$, let us define a coordinate y and a function $b(y)$ by

$$y^2 := r_{\text{ds}} - r, \quad b(y) := \sqrt{f_{\text{ds}}(r_{\text{ds}} - y^2)}. \quad (4.12)$$

We get $b'(y) := db(y)/dy = 2H^2 r y/b(y)$ which denotes

$$\lim_{y \rightarrow 0} b'(y) = 2H \lim_{y \rightarrow 0} \frac{y}{b(y)} = 2H \frac{1}{\lim_{y \rightarrow 0} b'(y)}. \quad (4.13)$$

This means $b'(0) = \sqrt{2H}$, and near the CEH, $f_{\text{ds}} \simeq [b(0) + b'(0)y]^2 = 2Hy^2$. Therefore the Euclidean metric near the CEH is

$$ds_E^2 \simeq \frac{2}{H} [y^2 d(H\tau)^2 + dy^2] + \frac{1}{H^2} d\Omega^2. \quad (4.14)$$

It is obvious that the Euclidean de Sitter space is regular at CEH if the imaginary time has the period β defined by

$$0 \leq \tau < \beta := \frac{2\pi}{H}. \quad (4.15)$$

Throughout our discussion, the imaginary time τ has the period β .

¹⁰If the signature of exponent in Eq.(4.7) is opposite, $W - X = \exp[-H(t - r^*)]$ and $W + X = -\exp[H(t + r^*)]$, then the direction of ∂_t becomes past pointing in the region I, and future pointing in the region II.

4.4.2 Euclidean action

Let us calculate the de Sitter's Euclidean action I_E defined by Eq.(2.7). To obtain I_E , we should specify the Lorentzian action I_L given in Eq.(2.9). The integral region \mathcal{M} in I_L is the Lorentzian region which forms the thermal equilibrium state constructed in the assumption dS-1. Therefore \mathcal{M} is given by $r_w < r < r_{\text{ds}}$, and its boundary $\partial\mathcal{M}$ is at the heat wall, $r = r_w$. There is another boundary at $r = r_{\text{ds}}$ in the Lorentzian region \mathcal{M} . However we do not need to consider it, because, as shown above, the points at $r = r_{\text{ds}}$ in *Euclidean* space do not form a boundary but are the regular points when the imaginary time τ has the period (4.15). Then the first fundamental form h_{ij} ($i, j = 0, 2, 3$) of $\partial\mathcal{M}$ in the static chart (in Lorentzian de Sitter spacetime) is

$$ds^2|_{r=r_w} = h_{ij} dx^i dx^j = -f_{\text{ds}}(r_w) dt^2 + r_w^2 d\Omega^2, \quad (4.16)$$

where $f_{\text{ds}}(r_w) = 1 - H^2 r_w^2$. Here, since \mathcal{M} is the region enclosed by the heat wall and CEH, the direction of unit normal vector \mathbf{n} to $\partial\mathcal{M}$ is pointing towards the “center” $r = 0$, which means $\mathbf{n} \propto -\partial_r$ (see Fig.2). Then the second fundamental form of $\partial\mathcal{M}$ in the static chart is

$$K_{ij} = -\sqrt{f_{\text{ds}}(r_w)} \text{diag.} [H^2 r_w, r_w, r_w \sin^2 \theta], \quad (4.17)$$

where diag. means the diagonal matrix form.

On the other hand, the second fundamental form $K_{ij}^{(\text{flat})}$ of a spherically symmetric timelike hypersurface $r = r_w$ in Minkowski spacetime is obtained by setting $H = 0$ in Eq.(4.17),

$$K_{ij}^{(\text{flat})} = -\text{diag.} [0, r_w, r_w \sin^2 \theta], \quad (4.18)$$

where the normal vector $\mathbf{n}^{(\text{flat})}$ to this surface is set to be $\mathbf{n}^{(\text{flat})} \propto -\partial_r$ as that in K_{ij} . For Minkowski spacetime $\mathcal{R} = 0$ and $\Lambda = 0$, and its Lorentzian action $I^{(\text{flat})}$ is expressed by only the surface term in Eq.(2.9),

$$I^{(\text{flat})} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} dx^3 \sqrt{\det h} K^{(\text{flat})}. \quad (4.19)$$

In applying this $I^{(\text{flat})}$ to Eq.(4.2) of working hypothesis dS-2, the integral element $\sqrt{\det h}$ in $I^{(\text{flat})}$ should be that of de Sitter spacetime, because the background spacetime on which the integral is calculated is the de Sitter spacetime.

From the above, applying the Wick rotation $t \rightarrow -i\tau$ to the Lorentzian action I_L of de Sitter spacetime, we obtain the Euclidean action I_E via Eqs.(2.7) and (4.2),

$$\begin{aligned} I_E &= \frac{3H^2}{8\pi} \int_{\mathcal{M}_E} dx_E^4 \sqrt{g_E} + \frac{1}{8\pi} \int_{\partial\mathcal{M}} dx_E^3 \sqrt{h_E} \left(K_E + \alpha_w K_E^{(\text{flat})} \right) \\ &= \frac{\pi}{H^2} \left(1 - 2H r_w f_{\text{ds}}(r_w) - 2\alpha_w H r_w \sqrt{f_{\text{ds}}(r_w)} \right), \end{aligned} \quad (4.20)$$

where the relation for de Sitter spacetime $\mathcal{R} = 4\Lambda = 12H^2$ is used in the first equality, Q_E is the quantity Q evaluated on the Euclidean de Sitter space, and \mathcal{M}_E is the Euclidean region expressed by $0 \leq \tau < \beta$, $r_w \leq r \leq r_{\text{ds}}$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. This I_E corresponds to $I_E[g_{Ecl}]$ in Eq.(2.16), which yields the partition function for the thermal equilibrium of spacetime with quantum fluctuations (whose effects are neglected in Eq.(2.15)). A remark on the limit $r_w \rightarrow 0$ of this I_E will be given in next subsection after specifying the form of α_w .

Let us make a comment for the micro-canonical ensemble. The action $I_E^{(\text{micro})}$ used in Eq.(4.1) is given by only the bulk term (first term) of I_L in Eq.(2.9). It yields for Euclidean de Sitter space,

$$I_E^{(\text{micro})} = \pi r_{\text{ds}}^2, \quad (4.21)$$

This is used in Eq.(4.1).

4.5 de Sitter Thermodynamics in the Canonical Ensemble

In this subsection, principal state variables of CEH are calculated successively, and the “thermodynamic consistency” is also shown explicitly.

4.5.1 Temperature

As explained in Eq.(2.11), the temperature T_{ds} of CEH is defined by the integral in imaginary time direction at the heat wall,

$$T_{\text{ds}} := \left[\int_0^\beta \sqrt{f_{\text{ds}}(r_w)} d\tau \right]^{-1} = \frac{H}{2\pi \sqrt{f_{\text{ds}}(r_w)}}, \quad (4.22)$$

where β is defined in Eq.(4.15) and $f_{\text{ds}}(r_w) = 1 - H^2 r_{\text{ds}}^2$. Note that, as implied by Eq.(4.6), $H/2\pi$ coincides with the Hawking temperature measured at $r = 0$ [12], and the factor $\sqrt{f_{\text{ds}}(r_w)}$ in Eq.(4.22) is the Tolman factor which expresses the gravitational redshift affecting the Hawking radiation propagating from CEH to the observer [26]. This T_{ds} is the temperature measured by the observer at heat wall.

4.5.2 Surface area and homothetic variation of the system

In the ordinary laboratory systems, the size of the system under consideration is its volume. For de Sitter spacetime, the volume can not be defined uniquely, since the choice of spatial slice in the region \mathcal{M} of $r_w < r < r_{\text{ds}}$ is not determined in a natural way. However, an area can be uniquely and naturally assigned to our system. It is the surface area A_{ds} of the heat wall.

The above discussion supports the assumption dS-2 in which A_{ds} is regarded as the state variable. Therefore, we adopt A_{ds} as the extensive state variable of system size,

$$A_{\text{ds}} := 4\pi r_w^2. \quad (4.23)$$

Here we have to note that, as explained in Appendix B in reference [1], the variation of system size is restricted to homothetic variations. For our system, the homothetic variation is the spherical variation due to the spherical symmetry.

4.5.3 Choice of α_w of working hypothesis dS-2

This subsection determines the form of α_w which appears in Eq.(4.20). As required in the working hypothesis dS-2, α_w is a dimensionless factor composed of the quantity determined at the heat wall. This implies that α_w is a function of the parameter,

$$x := H r_w, \quad (4.24)$$

which means $f_{\text{ds}}(r_w) = 1 - x^2$. This x is regarded as the dimensionless quantity determined at the heat wall. Then our Euclidean action I_E is expressed as

$$I_E = \frac{\pi}{H^2} \left(1 - 2x f_{\text{ds}}(r_w) - 2x \alpha_w(x) \sqrt{f_{\text{ds}}(r_w)} \right). \quad (4.25)$$

In order to determine $\alpha_w(x)$ so as to preserve the entropy-area law which is the equation of state verified already in the micro-canonical ensemble [15, 27], we need the free energy. The free energy F_{ds} of CEH is obtained via Eq.(2.16),

$$F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}}) := -\frac{1}{2H\sqrt{f_{\text{ds}}(r_w)}} \left(1 - 2x f_{\text{ds}}(r_w) - 2x \alpha_w(x) \sqrt{f_{\text{ds}}(r_w)} \right), \quad (4.26)$$

where H and x are regarded as functions of T_{ds} and A_{ds} . Then, by the assumption dS-3, the entropy S_{ds} of CEH is defined by,

$$S_{\text{ds}} := -\frac{\partial F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}})}{\partial T_{\text{ds}}} = -\frac{\partial_H F_{\text{ds}}}{\partial_H T_{\text{ds}}} = \frac{\pi}{H^2} D[\alpha_w(x)], \quad (4.27)$$

where

$$D[\alpha_w(x)] := -2x^2 f_{\text{ds}}(r_w)^{3/2} \frac{d\alpha_w}{dx} - (1 - 2x^3) f_{\text{ds}}(r_w) + x^2. \quad (4.28)$$

Therefore, to preserve the entropy-area law, $\alpha_w(x)$ has to be a solution of the first-order differential equation, $D[\alpha_w(x)] = 1$. Then we obtain

$$\alpha_w(x) = \left(\frac{1}{x} - 1 \right) \sqrt{f_{\text{ds}}(r_w)} + k_w, \quad (4.29)$$

where k_w is the integration constant. The value of k_w can not be determined at present. However, as will be shown at Eq.(4.37), by the existence of surface pressure at a limit $r_w \rightarrow 0$, we find k_w is zero,

$$k_w = 0. \quad (4.30)$$

Although the verification of this value is shown later, we proceed our calculations with setting $k_w = 0$ for simplicity of discussion. Then, adopting Eq.(4.30), the Euclidean action (4.20) is determined,

$$I_E = -\frac{\pi}{H^2} (1 - 2x^2). \quad (4.31)$$

It is conceptually important to consider the “small heat wall limit”, $r_w \rightarrow 0$. The Euclidean action I_E given in Eq.(4.31) takes the limit value $I_E \rightarrow -\pi r_{\text{ds}}^2$ as $r_w \rightarrow 0$. This value differs from $I_E^{(\text{micro})}$ in Eq.(4.21) by the negative signature. Here one may naively think that our I_E should coincide with $I_E^{(\text{micro})}$ at this limit. This naive requirement seems reasonable from the point of view of spacetime geometry, but is not necessarily reasonable from the point of view of statistical mechanics, because the coincidence of I_E with $I_E^{(\text{micro})}$ at the limit $r_w \rightarrow 0$ means that the micro-canonical ensemble is some limiting case of the canonical ensemble. In statistical mechanics, the micro-canonical ensemble is not some limiting case of the canonical ensemble. In de Sitter thermodynamics, the limit $r_w \rightarrow 0$ is just the case of an arbitrarily small heat wall and not the case without heat wall. Therefore, in statistical mechanical sense, there seems to be no reason to require that I_E coincides with $I_E^{(\text{micro})}$ at the limit $r_w \rightarrow 0$. (Concerning this discussion, see also the end of Sec.3 in which the limiting case of large heat bath for black hole thermodynamics is summarized.)

4.5.4 Free energy, entropy and second law

Previous subsection has given us the free energy F_{ds} and the entropy S_{ds} . The free energy is

$$F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}}) = \frac{1 - 2x^2}{2H \sqrt{f_{\text{ds}}(r_w)}}, \quad (4.32)$$

where x is defined in Eq.(4.24). As noted at the working hypothesis dS-1, F_{ds} should be regarded as a function of T_{ds} and A_{ds} . And the entropy is

$$S_{\text{ds}} = \frac{\pi}{H^2}. \quad (4.33)$$

Concerning the entropy, it should be noted that, exactly speaking, the second law is not a “theorem” proven by some other assumptions, but the basic assumption which can not be proven in the framework of thermodynamics. The best way to believe the second law is to “check” the statement of the law for as many processes as we can. For the black hole thermodynamics, the generalized second law is checked for some representative processes [4, 6]. For de Sitter thermodynamics, we need to check the generalized second law for as many processes as we can. However let us expect that the generalized second law holds also for the CEH in de Sitter spacetime, since our basic assumptions dS-1, 2, and 3 are the natural extension of consistent black hole thermodynamics.

4.5.5 Internal energy

The internal energy E_{ds} of CEH is defined by the argument of ordinary statistical mechanics,

$$E_{\text{ds}} := - \left. \frac{\partial \ln Z_{cl}}{\partial (1/T_{\text{ds}})} \right|_{A_{\text{ds}}=\text{const.}} = \frac{\partial (F_{\text{ds}}/T_{\text{ds}})}{\partial (1/T_{\text{ds}})} = F_{\text{ds}} + T_{\text{ds}} S_{\text{ds}} \quad (4.34)$$

where Eq.(2.16) is used at the second equality and the definition of S_{ds} in Eq.(4.27) is used at the third equality. The third equality, $E_{\text{ds}} = F_{\text{ds}} + T_{\text{ds}} S_{\text{ds}}$, is the Legendre transformation between $F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}})$ and E_{ds} . This implies that E_{ds} is a function of S_{ds} and A_{ds} , which is consistent with the ordinary thermodynamic argument that the internal energy is a function of extensive state variables. Then we get

$$E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}}) := \frac{1}{H} \sqrt{f_{\text{ds}}(r_w)}, \quad (4.35)$$

where H and x are regarded as functions of S_{ds} and A_{ds} via Eqs.(4.23), (4.24) and (4.33). The origin of internal energy E_{ds} will be discussed later in this subsection.

4.5.6 Surface pressure

As explained in Appendix B of reference [1], the conjugate state variable to A_{ds} is the surface pressure σ_{ds} defined by

$$\sigma_{\text{ds}} := -\frac{\partial F_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}})}{\partial A_{\text{ds}}} = -\frac{(\partial_H F_{\text{ds}})(\partial_{r_w} T_{\text{ds}}) - (\partial_{r_w} F_{\text{ds}})(\partial_H T_{\text{ds}})}{(\partial_H A_{\text{ds}})(\partial_{r_w} T_{\text{ds}}) - (\partial_{r_w} A_{\text{ds}})(\partial_H T_{\text{ds}})}, \quad (4.36)$$

where Eq.(2.19) is used at the second equality.

As mentioned at eq.(4.30), we determine the integration constant k_w here. To do so, we calculate σ_{ds} without setting k_w zero. From Eqs.(4.26) and (4.29), the free energy \tilde{F} with non-zero k_w is $\tilde{F} = F_{\text{ds}} + k_w r_w$, where F_{ds} is shown in Eq.(4.32). Then we obtain from Eq.(4.36),

$$\sigma_{\text{ds}} = \frac{H}{8\pi \sqrt{f_{\text{ds}}(r_w)}} - \frac{k_w}{8\pi r_w}. \quad (4.37)$$

It is obvious that, if $k_w \neq 0$, then the surface pressure diverges in the limit $r_w \rightarrow 0$. Hence, when we require the existence of finite σ_{ds} for thermal equilibrium states of CEH with arbitrarily small heat wall, it is natural to set $k_w = 0$. Then, with adopting this choice $k_w = 0$, we obtain

$$\sigma_{\text{ds}} = \frac{H}{8\pi \sqrt{f_{\text{ds}}(r_w)}} = \frac{1}{4} T_{\text{ds}}. \quad (4.38)$$

4.5.7 First and third laws

By definitions of S_{ds} and σ_{ds} ,

$$dF_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}}) = -S_{\text{ds}} dT_{\text{ds}} - \sigma_{\text{ds}} dA_{\text{ds}}. \quad (4.39)$$

Then the first law holds automatically via the Legendre transformation in Eq.(4.34),

$$dE_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}}) = d(F_{\text{ds}} + T_{\text{ds}} S_{\text{ds}}) = T_{\text{ds}} dS_{\text{ds}} - \sigma_{\text{ds}} dA_{\text{ds}}. \quad (4.40)$$

Next, to discuss the third law, note that T_{ds} is monotonically increasing as a function of r_w as shown by $\partial_{r_w} T_{\text{ds}}(H, r_w) = H^3 r_w / (2\pi f_{\text{ds}}(r_w)^{3/2}) > 0$. The minimum value of T_{ds} as a function of r_w is given at $r_w = 0$, $T_{\text{ds}}|_{r_w \rightarrow 0} = H/2\pi$. This denotes the zero-temperature state is achieved only by the process $H \rightarrow 0$ and $r_w \rightarrow 0$. However it is obvious from Eq.(4.35) that the infinite energy is required to realize the process $H \rightarrow 0$. The infinite energy supply is unphysical. Hence the third law holds in the sense that the zero-temperature state can not be achieved by any physical process.

4.5.8 Scaling law, Euler relation and Λ as a “hidden” variable

This subsection demonstrates the consistency of the scaling law, which is introduced in the assumption dS-2, with the other assumptions dS-1 and 3.

Let us consider the scaling of length size,

$$r_w \rightarrow \lambda r_w \quad , \quad r_{\text{ds}} \rightarrow \lambda r_{\text{ds}}. \quad (4.41)$$

The scaling of CEH radius denotes $H \rightarrow \lambda^{-1} H$. Then from Eqs.(4.22) and (4.38), we get the scaling of intensive variables,

$$T_{\text{ds}} \rightarrow \frac{1}{\lambda} T_{\text{ds}} \quad , \quad \sigma_{\text{ds}} \rightarrow \frac{1}{\lambda} \sigma_{\text{ds}} . \quad (4.42)$$

From Eqs.(4.23) and (4.33), the scaling of extensive variables is

$$A_{\text{ds}} \rightarrow \lambda^2 A_{\text{ds}} \quad , \quad S_{\text{ds}} \rightarrow \lambda^2 S_{\text{ds}} . \quad (4.43)$$

From Eqs.(4.32) and (4.35), we get the scaling of thermodynamic functions,

$$F_{\text{ds}} \rightarrow \lambda F_{\text{ds}} \quad , \quad E_{\text{ds}} \rightarrow \lambda E_{\text{ds}} . \quad (4.44)$$

Therefore we find that the scaling law of assumption dS-2 is consistent with the assumptions dS-1 and 3. Concerning this consistency, the definition of temperature in Eq.(2.11) should be emphasized. It is obvious that the temperature has the dimension of the inverse of length size by definition. Hence, the scaling law of assumption dS-2 is necessary to adopt Eq.(2.11) as the temperature which should be intensive.

Furthermore, to show a more robust consistency of the scaling law, recall that the internal energy is a function of S_{ds} and A_{ds} . Then the above scaling law implies

$$\lambda E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}}) = E_{\text{ds}}(\lambda^2 S_{\text{ds}}, \lambda^2 A_{\text{ds}}) . \quad (4.45)$$

This denotes that $E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}})$ is the homogeneous expression of degree 1/2. By the partial differential of this equation with respect to λ , we get the Euler relation,

$$\frac{1}{2} E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}}) = T_{\text{ds}} S_{\text{ds}} - \sigma_{\text{ds}} A_{\text{ds}} . \quad (4.46)$$

Note that the concrete functional forms of state variables is not used in deriving this Euler relation. The Euler relation (4.46) is obtained from the scaling behavior (4.45) and the differential relations implied by the first law, $T_{\text{ds}} \equiv \partial E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}})/\partial S_{\text{ds}}$ and $\sigma_{\text{ds}} \equiv -\partial E_{\text{ds}}(S_{\text{ds}}, A_{\text{ds}})/\partial A_{\text{ds}}$. On the other hand, we can check that the concrete forms of state variables T_{ds} , S_{ds} , σ_{ds} and A_{ds} obtained in previous subsections satisfy the relations (4.45) and (4.46). Hence the state variables obtained so far are completely consistent with the scaling law of assumption dS-2.

Finally in this subsection, recall that the cosmological constant Λ is regarded as an independent variable in the working hypothesis dS-1. It is obvious that the assumption dS-2, via the relation $\Lambda = 3 H^2$, excludes the “bare Λ ” from state variables, since Λ is neither intensive nor extensive. Hence the bare Λ can not be a “state variable” but a “hidden variable” in the consistent de Sitter thermodynamics.

4.5.9 Heat capacity and thermal stability

The thermodynamic consistency of our de Sitter canonical ensemble has been clearly checked so far. This subsection researches the thermal stability of CEH. The appropriate quantity to consider thermal stability is the heat capacity.

The representative heat capacity may be the heat capacity $C_{A_{\text{ds}}}$ at constant A_{ds} . Since A_{ds} depends only on r_w , $C_{A_{\text{ds}}}$ describes the response of temperature to the energy supply into CEH with fixing the position of observer r_w . $C_{A_{\text{ds}}}$ is defined by

$$C_{A_{\text{ds}}} := T_{\text{ds}} \frac{\partial S_{\text{ds}}(T_{\text{ds}}, A_{\text{ds}})}{\partial T_{\text{ds}}} = T_{\text{ds}} \frac{\partial_H S_{\text{ds}}}{\partial_H T_{\text{ds}}} , \quad (4.47)$$

where T_{ds} and A_{ds} are regarded as independent variables. By Eqs.(4.22) and (4.33), we get

$$C_{A_{\text{ds}}} = -\frac{2\pi}{H^2} f_{\text{ds}}(r_w) . \quad (4.48)$$

Obviously the heat capacity $C_{A_{\text{ds}}}$ is negative definite. When the energy is supplied to (extracted from) the CEH, then the temperature T_{ds} decreases (increases). This denotes the CEH is thermally unstable.

However, when we consider the constant Λ process as the physical one, this heat capacity $C_{A_{\text{ds}}}$ is the capacity for unphysical process, since $C_{A_{\text{ds}}}$ is defined by the derivative with respect to H as seen in Eq.(4.47). The thermal instability due to negative $C_{A_{\text{ds}}}$ seems to be unphysical.

When we consider the constant Λ process as the physical one, the heat capacity C_Λ at constant Λ is of interest. It is defined by

$$C_\Lambda := T_{\text{ds}} \frac{\partial S_{\text{ds}}(T_{\text{ds}}, H)}{\partial T_{\text{ds}}} = T_{\text{ds}} \frac{\partial_{r_w} S_{\text{ds}}}{\partial_{r_w} T_{\text{ds}}}, \quad (4.49)$$

where T_{ds} and Λ are regarded as independent variables. Then, since S_{ds} is independent of r_w , we find

$$C_\Lambda = 0. \quad (4.50)$$

Since C_Λ is not negative but *zero*, thermal equilibrium of CEH is not thermally unstable but thermally *marginal* stable for constant Λ process.

Since the constant Λ process means the variation of only the position of observer r_w with fixing the CEH radius $r_{\text{ds}} = \sqrt{3/\Lambda}$, the vanishing heat capacity (4.50) means that the observer's position r_w can change without a heat supply to the CEH. Here note that, Eq.(4.50) does not imply that changing r_w has no thermodynamic effect on the CEH. For example, changing r_w in constant Λ process gives rise to the change of surface pressure, $\partial \sigma_{\text{ds}} / \partial r_w \neq 0$. The vanishing heat capacity at constant Λ process (4.50) means simply the disappearance of heat supply in changing r_w with fixing Λ . This is a peculiar thermodynamic property of the CEH.

4.5.10 Surface compressibility and mechanical stability

Let us research the mechanical stability of our thermal equilibrium system of CEH. As explained in Appendix B of reference [1], the appropriate quantity to consider the mechanical stability may be the isothermal surface compressibility $\kappa_{T_{\text{ds}}}$ defined by $\kappa_{T_{\text{ds}}} := A_{\text{ds}}^{-1} \partial A_{\text{ds}}(T_{\text{ds}}, \sigma_{\text{ds}}) / \partial \sigma_{\text{ds}}$. However, since σ_{ds} is proportional to T_{ds} as shown in Eq.(4.38), the definition of $\kappa_{T_{\text{ds}}}$ becomes meaningless. Then, instead of $\kappa_{T_{\text{ds}}}$, let us consider the “isentropic” surface compressibility $\kappa_{S_{\text{ds}}}$ defined by

$$\kappa_{S_{\text{ds}}} := \frac{1}{A_{\text{ds}}} \frac{\partial A_{\text{ds}}(S_{\text{ds}}, \sigma_{\text{ds}})}{\partial \sigma_{\text{ds}}}, \quad (4.51)$$

where S_{ds} and σ_{ds} are regarded as independent variables. Since S_{ds} depends only on Λ as shown in Eq.(4.33), $\kappa_{S_{\text{ds}}}$ is equivalent to the surface compressibility at constant Λ , and therefore it seems to be the physical quantity. By Eqs.(4.23), (4.33) and (4.38), we get

$$\kappa_{S_{\text{ds}}} = \frac{1}{A_{\text{ds}}} \frac{\partial_{r_w} A_{\text{ds}}}{\partial_{r_w} \sigma_{\text{ds}}} = \frac{16 \pi f_{\text{ds}}(r_w)^{3/2}}{H x^2}. \quad (4.52)$$

Obviously $\kappa_{S_{\text{ds}}}$ is positive definite. When the surface A_{ds} increases, the surface pressure σ_{ds} also increases. If we take the same criterion of mechanical stability as York [7] (see the end of Appendix B of reference [1]), then the positivity of $\kappa_{S_{\text{ds}}}$ implies that our thermal equilibrium system is mechanically stable.

4.6 The Role of Cosmological Constant in de Sitter Thermodynamics

As mentioned in the working hypothesis dS-1, we regard Λ as a working variable to obtain thermodynamically consistent de Sitter canonical ensemble. The validity of working hypothesis dS-1 can be recognized simply by the following fact: The entropy $S_{\text{ds}} = 3\pi/\Lambda$ given in Eq.(4.33) depends only on Λ as already verified in the micro-canonical ensemble [15, 27], and consequently the definition of S_{ds} in Eq.(4.27) is expressed by using the derivatives of F_{ds} and T_{ds} with respect to Λ . The derivative with respect to Λ requires implicitly the variable Λ . Hence, in order to calculate the entropy in the canonical ensemble, it is necessary to adopt the working hypothesis dS-1. The following role of Λ is worth emphasizing;

- *The canonical ensemble of de Sitter spacetime constructs the “generalized” thermodynamics in which Λ behaves as a working variable, and the physical process is described by the constant Λ process.*

Finally in this section, let us discuss the origin of internal energy E_{ds} , which is related to Λ as explained below: For the first, recall the Schwarzschild thermodynamics formulated by York [7]. The internal energy E_{BH} in Schwarzschild thermodynamics is related to its mass parameter M by

$$M = E_{\text{BH}} - \frac{E_{\text{BH}}^2}{2r_w}, \quad (4.53)$$

where r_w is the outer-most radius of cavity shown in Fig.1. The second term $E_{\text{BH}}^2/(2r_w)$ can be interpreted as the self-gravitational potential energy of black hole. Then E_{BH} is interpreted as the “bare” mass energy of the *black hole in cavity*, while M is the “net” mass energy including the self-gravitational potential. It seems reasonable to consider that the origin of internal energy E_{BH} is the mass of black hole. The mass M as the origin of energy E_{BH} can be clearly exhibited in the large cavity limit $r_w \rightarrow \infty$. In this limit we have $E_{\text{BH}}|_{r_w \rightarrow \infty} = M$, which manifestly shows that E_{BH} is originated from M .

Then turn our discussion to de Sitter thermodynamics. Let us consider the small heat wall limit $r_w \rightarrow 0$, which seems to correspond to the large cavity limit in Schwarzschild thermodynamics, since the heat wall is most distant from CEH. In this limit we have

$$\lim_{r_w \rightarrow 0} E_{\text{ds}} = \frac{1}{H}. \quad (4.54)$$

This may show that the origin of internal energy E_{ds} is the cosmological constant $\Lambda (= 3H^2)$. In the framework of classical general relativity, the de Sitter spacetime is a vacuum spacetime which includes no energy source. However, in the de Sitter thermodynamics which includes essentially the quantum gravitational effects, Λ may be interpreted as a kind of energy source which is responsible to the energy E_{ds} . Also Λ may be responsible to the entropy S_{ds} .

5 Limit of Entropy-Area Law for Multi-Horizon Spacetimes

Let us proceed to the discussion on the universality of entropy-area law for black hole event horizon (BEH) and cosmological event horizon (CEH) in SdS spacetime. As discussed in Sec.1, some reasonable evidence of the breakdown of entropy-area law is going to be revealed in this section, which indicate the following:

- Thermal equilibrium of individual horizon in multi-horizon spacetime is just a necessary condition of entropy-area law.
- The necessary and sufficient condition of entropy-area law is the thermal equilibrium of the total system composed of several horizons in which the net energy flow among horizons disappears.

5.1 Summary of Schwarzschild-de Sitter Geometry

In order to prepare some quantities used in following analysis, let us summarize the Lorentzian SdS spacetimes. The metric of SdS spacetime in the *static chart* is

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (5.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element on the unit two-sphere, and

$$f(r) := 1 - \frac{2M}{r} - H^2 r^2, \quad 3H^2 := \Lambda, \quad (5.2)$$

where M is the mass parameter of black hole and Λ is the cosmological constant. The Penrose diagram of SdS spacetime is shown in Fig.3, and the static chart covers the region I.

An algebraic equation $f(r) = 0$ has one negative root and two positive roots. The smaller and larger positive roots are, respectively, the radius of BEH r_b and that of CEH r_c . The notion of CEH is observer dependent and the CEH at r_c is associated with the observer going towards the temporal future infinity

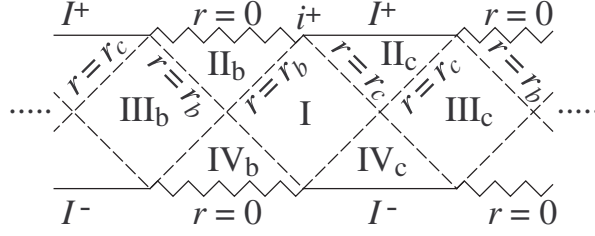


Figure 3: Penrose diagram of SdS spacetime. I^\pm are the future/past null infinity, i^+ is the future temporal infinity. BEH is at $r = r_b$, and CEH at $r = r_c$. Spacetime singularity is at $r = 0$. Static chart covers the region I. Semi-global black hole chart covers the regions I, II_b , III_b and IV_b . Semi-global cosmological chart covers the regions I, II_c , III_c and IV_c . The maximal extension is obtained by connecting the two semi-global charts alternately.

i^+ in region I [12]. The equation $f(r) = 0$ is rearranged to $4\tilde{r}^3 - 3\tilde{r} + \sqrt{27}MH = 0$, where $\tilde{r} := \sqrt{3}Hr/2$. Then via a formula, $\sin\theta = -4\sin^3(\theta/3) + 3\sin(\theta/3)$, we get

$$r_b = \frac{2}{\sqrt{3}H} \sin\left(\frac{\alpha}{3}\right) \quad , \quad r_c = \frac{2}{\sqrt{3}H} \sin\left(\frac{\alpha + 2\pi}{3}\right) \quad , \quad (5.3)$$

where α is defined by, $\sin\alpha := \sqrt{27}MH$. The existence condition of BEH and CEH is $0 < \sqrt{27}MH < 1$. This is equivalent to, $0 < \alpha < \pi/2$, which means

$$2M < r_b < 3M < \frac{1}{\sqrt{3}H} < r_c < \frac{1}{H} \quad . \quad (5.4)$$

This denotes that r_b is larger than the Schwarzschild radius $2M$ and r_c is smaller than the de Sitter's CEH radius H^{-1} .

SdS spacetime has a timelike Killing vector $\xi := N\partial_t$, where N is a normalization constant [28]. This ξ becomes null at BEH and CEH. This means those horizons are the Killing horizons of ξ . The surface gravity of BEH κ_b and that of CEH κ_c are defined by the equations, $\nabla_\xi \xi^\mu|_{r=r_b} = \kappa_b \xi^\mu|_{r=r_b}$, $\nabla_\xi \xi^\mu|_{r=r_c} = \kappa_c \xi^\mu|_{r=r_c}$. The surface gravity depends on N . Throughout this section we take the normalization $N = 1$ for BEH, and $N = -1$ for CEH to make κ_c positive¹¹. Then the surface gravities become equal to the absolute value $|(1/2)df(r)/dr|$ at each Killing horizon,

$$\begin{aligned} \kappa_b &= \frac{H^2}{2r_b} (r_c - r_b) (2r_b + r_c) = \frac{1}{2r_b} (1 - 3H^2 r_b^2) \quad , \\ \kappa_c &= \frac{H^2}{2r_c} (r_c - r_b) (r_b + 2r_c) = \frac{1}{2r_c} (3H^2 r_c^2 - 1) \quad , \end{aligned} \quad (5.5)$$

where Eq.(5.3) is used in the second equality in each equation. From the inequality $r_b < r_c$ in Eq.(5.4), we get

$$\kappa_b > \kappa_c \quad . \quad (5.6)$$

This implies the Hawking temperature of BEH is higher than that of CEH, which will be verified by Eqs.(5.42) and (5.62).

For later use, let us show some differentials,

$$\frac{\partial r_b}{\partial M} = \frac{2}{1 - 3H^2 r_b^2} \quad , \quad \frac{\partial r_b}{\partial H} = -\frac{r_b}{H} + \frac{M}{H} \frac{\partial r_b}{\partial M} \quad , \quad (5.7a)$$

$$\frac{\partial r_c}{\partial M} = -\frac{2}{3H^2 r_c^2 - 1} \quad , \quad \frac{\partial r_c}{\partial H} = -\frac{r_c}{H} + \frac{M}{H} \frac{\partial r_c}{\partial M} \quad , \quad (5.7b)$$

¹¹Even if $N = 1$ for CEH, we can keep consistency of our analysis by changing the signature of κ_c appropriately.

and

$$\frac{\partial \kappa_b}{\partial M} = -\frac{1}{r_b^2} \frac{1 + 3H^2 r_b^2}{1 - 3H^2 r_b^2}, \quad \frac{\partial \kappa_b}{\partial H} = \frac{1 - 3H^2 r_b^2}{2H r_b} + \frac{M}{H} \frac{\partial \kappa_b}{\partial M}, \quad (5.7c)$$

$$\frac{\partial \kappa_c}{\partial M} = -\frac{1}{r_b^2} \frac{3H^2 r_c^2 + 1}{3H^2 r_c^2 - 1}, \quad \frac{\partial \kappa_c}{\partial H} = \frac{3H^2 r_c^2 - 1}{2H r_c} + \frac{M}{H} \frac{\partial \kappa_c}{\partial M}, \quad (5.7d)$$

where we used a formula, $\cos \theta = 4 \cos^3(\theta/3) - 3 \cos(\theta/3)$, and the differentials, $\partial_M \alpha = \sqrt{27}H/\cos \alpha$, and $\partial_H \alpha = \sqrt{27}M/\cos \alpha$, obtained from the definition of α , $\sin \alpha := \sqrt{27}MH$.

The metric in *semi-global black hole chart* is given by the coordinate transformation from (t, r, θ, φ) to $(\eta_b, \chi_b, \theta, \varphi)$:

$$\eta_b - \chi_b := -e^{-\kappa_b(t-r^*)}, \quad \eta_b + \chi_b := e^{\kappa_b(t+r^*)}, \quad (5.8)$$

where $dr^* := dr/f(r)$ which means

$$2r^* = \ln \left| \frac{r}{r_b} - 1 \right|^{1/\kappa_b} - \ln \left| 1 - \frac{r}{r_c} \right|^{1/\kappa_b} + \ln \left| \frac{r}{r_b + r_c} + 1 \right|^{1/\kappa_c - 1/\kappa_b}. \quad (5.9)$$

We get by this transformation,

$$ds^2 = \Upsilon_b(r) [-d\eta_b^2 + d\chi_b^2] + r^2 d\Omega^2, \quad (5.10)$$

where

$$\Upsilon_b(r) := \frac{2M}{\kappa_b^2 r} \left(1 - \frac{r}{r_c} \right)^{1+\kappa_b/\kappa_c} \left(\frac{r}{r_b + r_c} + 1 \right)^{2-\kappa_b/\kappa_c}. \quad (5.11)$$

The transformation (5.8) implies the range of coordinates, $-\chi_b < \eta_b < \chi_b$ and $0 < \chi_b$, which covers the region I in Fig.3. By extending to the range, $-\infty < \eta_b < \infty$ and $-\infty < \chi_b < \infty$, the semi-global black hole chart covers the regions I, II_b, III_b and IV_b in Fig.3. In these regions we find $\Upsilon_b > 0$ since $r < r_c$.

The metric in *semi-global cosmological chart* is given by the coordinate transformation from (t, r, θ, φ) to $(\eta_c, \chi_c, \theta, \varphi)$:

$$\eta_c - \chi_c := e^{\kappa_c(t-r^*)}, \quad \eta_c + \chi_c := -e^{-\kappa_c(t+r^*)}, \quad (5.12)$$

where r^* is given in Eq.(5.9). By this transformation we get

$$ds^2 = \Upsilon_c(r) [-d\eta_c^2 + d\chi_c^2] + r^2 d\Omega^2, \quad (5.13)$$

where

$$\Upsilon_c(r) := \frac{2M}{\kappa_c^2 r} \left(\frac{r}{r_b} - 1 \right)^{1+\kappa_c/\kappa_b} \left(\frac{r}{r_b + r_c} + 1 \right)^{2-\kappa_c/\kappa_b}. \quad (5.14)$$

The transformation (5.12) implies the range of coordinates, $\chi_c < \eta_c < -\chi_c$ and $\chi_c < 0$, which covers the region I in Fig.3. By extending to the range, $-\infty < \eta_c < \infty$ and $-\infty < \chi_c < \infty$, the semi-global cosmological chart covers the regions I, II_c, III_c and IV_c in Fig.3. In these regions we find $\Upsilon_c > 0$ since $r_b < r$.

The maximally extended SdS spacetime is obtained by connecting the two semi-global charts alternately as shown in Fig.3.

5.2 Minimal Assumptions for Schwarzschild-de Sitter Thermodynamics

As mentioned in Sec.2.1, we need to specify the appropriate state variables before culculating the Euclidean action. To do so, we introduce the minimal set of assumptions with referring to York's Schwarzschild thermodynamics [7]. As reviewed in Sec.3, there are three key points in Schwarzschild thermodynamics from which we can learn how to ensure the ‘‘thermodynamic consistency’’ in SdS thermodynamics.

Here, before considering SdS spacetime, we must comment on Anti-de Sitter (AdS) black holes [9]. AdS black hole thermodynamics has a conceptual difference from the other black hole thermodynamics. The difference appears, for example, in the definition of temperature. While the temperatures in

asymptotic flat black hole and de Sitter thermodynamics include the *Tolman factor* [7, 8, 1, 26], the temperature assigned to AdS black hole [9] does not include the Tolman factor, where the Tolman factor [26] expresses the gravitational redshift affecting the Hawking radiation propagating from horizon to observer (see for example Eq.(5.42) shown later or the key point 3 of Schwarzschild thermodynamics in Sec.3). The temperature in AdS black hole thermodynamics can not be measured by a thermometer of the physical observer who are outside the black hole. This implies that the state variables in AdS black hole thermodynamics are defined not by the observer outside the black hole, but defined just on the black hole event horizon on which no physical observer can rest. In this article we do not refer to AdS black hole thermodynamics, since it seems to be preferable to expect that state variables are defined according to a physical observer.

5.2.1 Zeroth law and independent variables

We will construct two thermal equilibrium systems for BEH and CEH, but place only one observer who can measure the state variables of BEH and CEH. Such observer is in the region I, $r_b < r < r_c$ (see Fig.3). However, as mentioned at Eq.(5.6), Hawking temperature of BEH is higher than that of CEH. This temperature difference implies that, when the region I constitutes one thermodynamic system, the thermodynamic state of region I is in a non-equilibrium state. Therefore, by dividing the region I into two regions, we construct *two* thermal equilibrium systems for BEH and CEH individually which are measured by the same observer. To do so, we adopt the following assumption as the zeroth law:

Assumption SdS – 1 (Zeroth law) *Two thermal “equilibrium” systems for BEH and CEH in SdS spacetime are constructed by the following three steps:*

1. *Place a spherically symmetric thin wall at $r = r_w$ in the region I, $r_b < r_w < r_c$, as shown in Fig.4. This wall has negligible mass, and reflects perfectly Hawking radiation coming from each horizon. We call this wall the “heat wall” hereafter. The BEH side of heat wall is regarded as a “heat bath” of Hawking temperature of BEH due to the reflected Hawking radiation. Also the CEH side of heat wall is regarded as a heat bath of Hawking temperature of CEH.*
2. *The region D_b enclosed by BEH and heat wall, $r_b < r < r_w$, is filled with Hawking radiation emitted by BEH and reflected by heat wall, and forms a thermal equilibrium system for BEH. Similarly the region D_c enclosed by CEH and heat wall, $r_w < r < r_c$, forms a thermal equilibrium system for CEH. Hence we have “two” thermal equilibrium systems separated by the heat wall. In the statistical mechanical sense, these two thermal equilibrium systems are described by the canonical ensemble, since those systems have a contact with the heat wall.*
3. *Set our observer at the heat wall. When the observer is at the BEH side of heat wall, the observer can measure the state variables of thermal equilibrium system for BEH. The same is true of CEH. Then the state variables of two thermal equilibrium systems are defined by the quantities measured by the observer at heat wall.*

Note that the two thermal equilibrium systems constructed in this assumption have already been used by Gibbons and Hawking [12] to calculate Hawking temperatures of BEH and CEH. Also the above step 3, which gives a criterion of defining state variables, has already been adopted in the consistent thermodynamics of single-horizon spacetimes [7, 8, 1]. This assumption is a simple extension of the key point 1 of Schwarzschild thermodynamics shown in Sec.3.

It is expected that state variables of the thermal equilibrium system for BEH depend on BEH radius r_b and/or BEH surface gravity κ_b . Similarly, state variables of CEH depend on r_c and/or κ_c . These imply that the state variables of BEH and CEH depend on M and Λ , since the horizon radii and surface gravities depend on M and Λ via Eqs.(5.3) and (5.5). Furthermore, by the step 3 in assumption SdS-1, there should be r_w -dependence in state variables of BEH and CEH, since the observer is at $r = r_w$. Therefore the state variables of BEH and CEH depend on three parameters M , Λ and r_w .

The existence of three parameters (M, Λ, r_w) may imply that the CEH is regarded as a source of external gravitational field which affects the thermodynamic state of BEH. Also, BEH is a source of

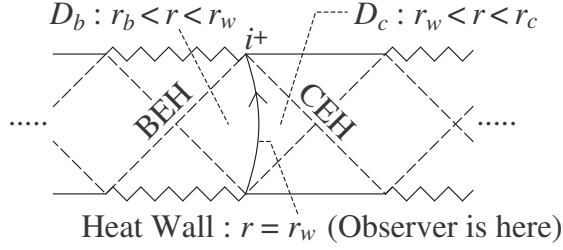


Figure 4: Our thermal equilibrium systems for BEH (D_b) and CEH (D_c). State variables of them are defined at the heat wall.

external gravitational field affecting the thermodynamic state of CEH. Here it is instructive to compare *qualitatively* our two thermal equilibrium systems of horizon with a magnetized gas. The gas consists of molecules possessing a magnetic moment, and its thermodynamic state is affected by an external magnetic field. The qualitative correspondence between the magnetized gas and our thermal equilibrium systems of horizons is described as follows; the gas corresponds to the system D_b (D_c), and the external magnetic field corresponds to the external gravitational field produced by CEH (BEH). Then, what we should emphasize is the following fact of the magnetized gas: When the gas is enclosed in a container of volume V_{gas} and an external magnetic field \vec{H}_{ex} is acting on the gas, the free energy F_{gas} of the gas is expressed as a function of three independent state variables (see for example §52, 59 and 60 in Landau and Lifshitz [23]),

$$F_{\text{gas}} = F_{\text{gas}}(T_{\text{gas}}, V_{\text{gas}}, \vec{H}_{\text{ex}}), \quad (5.15)$$

where T_{gas} is the temperature of the gas. According to this fact of the magnetized gas, it is reasonable for our two thermal equilibrium systems of horizons to require that the free energies are also functions of three independent state variables. For the BEH, the free energy F_b is

$$F_b = F_b(T_b, A_b, X_b), \quad (5.16)$$

where T_b is the temperature of BEH, A_b is the state variable of system size, and X_b is the state variable which represents the effect of CEH's gravity on the BEH. And for the CEH, the free energy F_c is

$$F_c = F_c(T_c, A_c, X_c), \quad (5.17)$$

where T_c is the temperature of CEH, A_c is the state variable of system size and X_c is the state variable which represents the effect of BEH's gravity on the CEH. Indeed, it will be proven in Secs. 5.4 and 5.5 that the thermodynamic consistency never hold unless the free energies are functions of three independent variables as shown in Eqs.(5.16) and (5.17).

Now we recognize that, because free energies are functions of three independent variables (as will be verified in Secs. 5.4 and 5.5), the following working hypothesis is needed:

Working Hypothesis SdS – 1 (Three independent variables) *To ensure the thermodynamic consistency of our thermal equilibrium systems constructed in assumption SdS-1, we have to regard the cosmological constant Λ as an independent working variable. Then the three quantities (M, Λ, r_w) are regarded as independent variables, and consequently the free energies F_b and F_c of our thermal equilibrium systems are functions of three independent state variables as shown in Eqs.(5.16) and (5.17). On the other hand, when we regard the non-variable Λ as the physical situation, it is obtained by the “constant Λ process” in the consistent thermodynamics for BEH and CEH which are constructed with regarding Λ as a working variable.*

This working hypothesis will be verified in Secs. 5.4 and 5.5, and we can not preserve thermodynamic consistency without regarding Λ as an independent working variable. Therefore, as already noticed for de Sitter canonical ensemble in Sec.4.6, the special role of cosmological constant is recognized:

- *Thermal equilibrium states of event horizon with positive Λ may construct the “generalized” thermodynamics in which Λ behaves as a working variable and the physical process is described by the constant Λ process.*

5.2.2 Scaling law and system size

In thermodynamics of ordinary laboratory systems, all state variables are classified into two categories, *extensive* state variables and *intensive* ones. The criterion of this classification is the scaling behavior of state variables under the scaling of system size. However, as explained by the key point 2 of Schwarzschild thermodynamics in Sec.3, the state variables in thermodynamics of single-horizon spacetimes [7, 8, 1] have its own peculiar scaling behavior classified into three categories, and the state variable of system size is not a volume but the surface area of heat bath. Although the scaling behavior differs from that in thermodynamics of ordinary laboratory systems, the peculiar scaling behavior in thermodynamics of single-horizon spacetimes retains the thermodynamic consistency as explained in Secs.3 and 4. Then, we assume that the key point 2 of Schwarzschild thermodynamics is simply extended to our two thermal equilibrium systems constructed in assumption SdS-1:

Assumption SdS – 2 (Scaling law and system size) *All state variables of our thermal equilibrium systems are classified into three categories; extensive variables, intensive variables and thermodynamic functions, and satisfy the following scaling law: When the length size L (e.g. horizon radius) is scaled as $L \rightarrow \lambda L$ with an arbitrary scaling rate $\lambda (> 0)$, then the extensive variables X (e.g. system size) and intensive variables Y (e.g. temperature) are scaled respectively as $X \rightarrow \lambda^2 X$ and $Y \rightarrow \lambda^{-1} Y$, while the thermodynamic functions Φ (e.g. free energy) are scaled as $\Phi \rightarrow \lambda \Phi$. This implies that the thermodynamic system size of our thermal equilibrium systems should have the areal dimension, since the system size is extensive in thermodynamic argument. Then we assume that the surface area of heat wall, $A := 4\pi r_w^2$, behaves as the consistent extensive variable of system size for our thermal equilibrium systems for BEH and CEH. This denotes to set $A_b = A_c = A$ in Eqs.(5.16) and (5.17).*

Accepting this assumption, the length size scaling in our thermal equilibrium systems for BEH and CEH should be specified. Here recall that, due to the working hypothesis SdS-1, the fundamental independent variables in our thermal equilibrium systems are M , Λ and r_w . Therefore the fundamental length size scaling for our thermal equilibrium systems is composed of the following three scalings;

$$M \rightarrow \lambda M \quad , \quad H \rightarrow \frac{1}{\lambda} H \quad , \quad r_w \rightarrow \lambda r_w \quad , \quad (5.18)$$

where $3H^2 = \Lambda$, and $\lambda (> 0)$ is an arbitrary scaling rate. The extensivity and intensivity of each state variable of our thermal equilibrium systems should be defined under these fundamental length size scalings as explained in assumption SdS-2.¹²

5.2.3 Euclidean action method and how to obtain state variables

We need to specify how to get the state variables. As noted in the step 2 in assumption SdS-1, thermodynamics of our two thermal equilibrium systems should be constructed in the canonical ensemble. Therefore we use the Euclidean action method which is the technique to calculate the partition function of quantum gravity [13]. Indeed, the Euclidean action method has already made successes to obtain the partition function of canonical ensemble for the thermodynamics of single-horizon spacetimes [7, 8, 9, 1]. The key point 3 in Sec.3 explains how to use the Euclidean action and to define state variables for Schwarzschild thermodynamics. Then, referring to the key point 3, we adopt the following assumption:

¹² As explained in Appendix B of paper [1], when we regard $A := 4\pi r_w^2$ as a state variable of system size, the scaling of system size should be restricted to the homothetic one, which is the spherical scaling due to the spherical symmetry of SdS spacetime. The fundamental length size scaling (5.18) is consistent with this restriction. See Appendix B of paper [1] for details of such restriction.

Assumption SdS – 3 (State variables and Euclidean action method) *Euclidean actions I_{Eb} and I_{Ec} of our two thermal equilibrium systems yield the partition functions of canonical ensembles by Eq.(2.15). And the free energies F_b and F_c are defined by Eq.(2.16), where the temperatures are defined by Eq.(2.11). Then, once F_b and F_c are determined, all state variables of BEH and CEH are defined from F_b and F_c as for thermodynamics of ordinary laboratory systems. For example, BEH entropy S_b is defined by $S_b := -\partial F_b / \partial T_b$, where T_b is the temperature of BEH.*

When we use the Euclidean action method, it is necessary to specify the integration constant I_{sub} . It is natural to require that our thermal equilibrium systems for BEH and CEH should reproduce, respectively, the Schwarzschild thermodynamics in the limit $\Lambda \rightarrow 0$ and the de Sitter thermodynamics in the limit $M \rightarrow 0$. Then the following working hypothesis is naturally required:

Working Hypothesis SdS – 2 (Integration constants in Euclidean action) *For the thermal equilibrium system for BEH, the integration constant in Euclidean action is determined with referring to Schwarzschild canonical ensemble formulated by York [7]. For the thermal equilibrium system for CEH, the integration constant in Euclidean action is determined with referring to de Sitter canonical ensemble [1] formulated in Sec.4.*

We introduce this working hypothesis as if this is a statement separated from the assumption SdS-3. However the determination of integration constant accompanies necessarily the Euclidean action method. The working hypothesis SdS-2 is regarded as a part of the assumption SdS-3.

5.2.4 Effects of external gravitational fields

By the assumption SdS-3 together with the working hypothesis SdS-2, the concrete functional form of free energies F_b and F_c can be determined as functions of three independent working variables (M, Λ, r_w) . However, since the form of the state variables X_b and X_c have not been specified yet, we can not rearrange F_b and F_c to functions of independent *state variables*, (T_b, A, X_b) and (T_c, A, X_c) .

As explained at Eqs.(5.16) and (5.17), the CEH (BEH) is regarded as the source of external gravitational field which affects the thermodynamic state of BEH (CEH). This means that the state variables X_b and X_c represent, respectively, the thermodynamic effect of CEH's gravity on BEH and that of BEH's gravity on CEH. Then it is reasonable to expect that X_b depends on the quantity characterizing the CEH's gravity, and X_c depends on the quantity characterizing the BEH's gravity. Moreover, due to the step 3 in assumption SdS-1, X_b and X_c should be measurable for the observer at r_w . Then, we can offer two candidates for the pair of *dimensionless* characteristic quantities of BEH's and CEH's gravities;

- First candidate pair consists of $\kappa_b r_w$ and $\kappa_c r_w$, where $\kappa_b r_w$ is for BEH's gravity and $\kappa_c r_w$ is for CEH's gravity. This pair means that both of BEH's and CEH's gravities are characterized by three quantities (M, Λ, r_w) , since κ_b and κ_c depend on M and Λ .
- Second candidate pair consists of M/r_w and $H r_w$, where M/r_w is for BEH's gravity and $H r_w$ is for CEH's gravity. This pair means that the BEH's gravity is characterized by (M, r_w) , and the CEH's gravity is characterized by (Λ, r_w) .

Here, purely logically, we can consider the “inverse” pair of second one, where $H r_w$ is for BEH's gravity and M/r_w is for CEH's gravity. This means that the BEH's gravity is characterized by (Λ, r_w) , and the CEH's gravity is characterized by (M, r_w) . However this is physically unacceptable, since we do not expect that the BEH does not depend on M and the CEH does not depend on Λ . Therefore, the reasonable candidates for the pair of characteristic quantities of BEH's and CEH's gravities are the two candidates listed above. Then, X_b should be a function of (κ_c, r_w) or (H, r_w) , and X_c should be a function of (κ_b, r_w) or (M, r_w) .

On the other hand, as will be mathematically verified in Secs.5.4 and 5.5, the state variables X_b and X_c are the extensive variables and proportional to r_w^2 . The proportionality to r_w^2 is consistent with the scaling law of extensive variables denoted in assumption SdS-2. And, according to the previous paragraph, the factor of proportionality should be a function of the characteristic quantity of BEH's or CEH's gravity.

Although the verification of the extensivity of X_b and X_c are shown later, we accept it in the following assumption SdS-4 for the simplicity of our discussion.

From the above, it is reasonable to adopt the following assumption:

Assumption SdS – 4 (Extensive variable of “external field”) *The state variables X_b and X_c in Eqs.(5.16) and (5.17) are the extensive variables. (This will be verified in Secs.5.4 and 5.5). Then, there are two candidates for the functional forms of X_b and X_c . One of them is based on the quantities $(\kappa_b r_w, \kappa_c r_w)$:*

$$X_b = r_w^2 \Psi_b(\kappa_c r_w) \quad , \quad X_c = r_w^2 \Psi_c(\kappa_b r_w) \quad , \quad (5.19)$$

where Ψ_b and Ψ_c are arbitrary functions of single argument, whose functional forms are not specified at present. Another candidate of X_b and X_c is based on the quantities $(M/r_w, H r_w)$:

$$X_b = r_w^2 \Psi_b(H r_w) \quad , \quad X_c = r_w^2 \Psi_c(M/r_w) \quad . \quad (5.20)$$

At least for the present author, no criterion to choose one of these candidates is found, and the way for determining the functional forms of Ψ_b and Ψ_c are also unknown. An obvious constraint on Ψ_b and Ψ_c is that they never be constant to make X_b and X_c independent of the state variable of system size $A := 4\pi r_w^2$.

In Sec.6, we will make some comments on the issue which of Eqs.(5.19) and (5.20) is valid. Those comments will suggest that Eq.(5.19) may be appropriate, but we do not have mathematical verification to choose Eq.(5.19) as the general form of X_b and X_c . Therefore, to retain the logical strictness of this article, we list the two possibilities (5.19) and (5.20) in the assumption SdS-4.

From the above, we recognize that the minimal set of assumptions for “consistent” thermodynamics of our two thermal equilibrium systems should be composed of four assumptions. However, the determination of functions Ψ_b and Ψ_c remains as a future task and we can not find concrete functional forms of them. Although the state variables X_b and X_c are not specified in this article, the existence of them enables us to examine the validity of entropy-area law in SdS spacetime as shown in Secs. 5.4 and 5.5.

5.3 Euclidean Actions for Two Horizons

Referring to assumption SdS-3 and working hypothesis SdS-2, we calculate Euclidean actions for the two thermal equilibrium systems for BEH and CEH constructed in assumption SdS-1.

5.3.1 Euclidean action for BEH

Euclidean space of thermal equilibrium system for BEH is obtained by the Wick rotations $t \rightarrow -i\tau$ in the static chart and $\eta_b \rightarrow -i\omega_b$ in the semi-global black hole chart. These Wick rotations are equivalent, because the coordinate transformation (5.8), $\eta_b = e^{\kappa_b r^*} \sinh(\kappa_b t)$, implies that the imaginary time ω_b in the semi-global chart is defined by $\omega_b := e^{\kappa_b r^*} \sin(\kappa_b \tau)$, where τ is the imaginary time in the static chart. Euclidean metric in the static chart is

$$ds_E^2 = f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad . \quad (5.21)$$

Euclidean metric in the semi-global black hole chart is

$$ds_E^2 = \Upsilon_b(r) [d\omega_b^2 + d\chi_b^2] + r^2 d\Omega^2 \quad , \quad (5.22)$$

where Υ_b is defined in Eq.(5.11). About the semi-global chart, we get from the coordinate transformation (5.8),

$$\omega_b^2 + \chi_b^2 = \left(\frac{r}{r_b} - 1\right) \left(1 - \frac{r}{r_c}\right)^{-\kappa_b/\kappa_c} \left(\frac{r}{r_b + r_c} + 1\right)^{-1+\kappa_b/\kappa_c} \quad . \quad (5.23)$$

Then, because the thermal equilibrium system for BEH is the region D_b , $r_b < r < r_w$, in Lorentzian SdS spacetime, we find the topology of Euclidean space of thermal equilibrium system for BEH is $D^2 \times S^2$.

Because $\omega_b^2 + \chi_b^2 = 0$ at $r = r_b$, the center of D^2 -part is at the BEH $r = r_b$, and the boundary of D^2 -part is at the heat wall $r = r_w$. The topology of heat wall boundary is $S^1 \times S^2$, where S^1 is along the τ -direction.

Because the Lorentzian SdS spacetime is regular at r_b , the Euclidean space is also regular at r_b . To examine the regularity of Euclidean space at r_b , we make use of the static chart (5.21). Let us define a coordinate y_b and a function $\gamma(y_b)$ by

$$y_b^2 := r - r_b \quad , \quad \gamma(y_b) := \sqrt{f(r_b + y_b^2)} . \quad (5.24)$$

We get $\gamma'(y_b) := d\gamma(y_b)/dy_b = [y_b/\gamma(y_b)] df(r)/dr$, from which we find

$$\lim_{y_b \rightarrow 0} \gamma'(y_b) = \frac{dC}{dr} \bigg|_{r_b} \lim_{y_b \rightarrow 0} \frac{y_b}{\gamma(y_b)} = 2\kappa_b \frac{1}{\lim_{y_b \rightarrow 0} \gamma'(y_b)} . \quad (5.25)$$

This means $\gamma'(0) = \sqrt{2\kappa_b}$, and near the BEH, $f \simeq [\gamma(0) + \gamma'(0)x]^2 = 2\kappa_b y_b^2$. Therefore the Euclidean metric near BEH is

$$ds_E^2 \simeq \frac{2}{\kappa_b} [y_b^2 d(\kappa_b \tau)^2 + dy_b^2] + r^2 d\Omega^2 . \quad (5.26)$$

It is obvious that the Euclidean space is regular at BEH if the imaginary time has the period β_b defined by

$$0 \leq \tau < \beta_b := \frac{2\pi}{\kappa_b} . \quad (5.27)$$

Throughout our discussion, τ has the period β_b in the Euclidean space of thermal equilibrium system for BEH.

Now let us proceed to the calculation of the Euclidean action I_{Eb} of thermal equilibrium system for BEH via Eq.(2.7), where the definition of Lorentzian action I_L is given in Eq.(2.9). Following the working hypothesis SdS-2, we use the same integration constant I_{sub} as Schwarzschild canonical ensemble [13, 7], which gives us

$$I_{\text{sub}} := -I_L^{(\text{flat})} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} dx^3 \sqrt{\det h} K^{(\text{flat})} , \quad (5.28)$$

where $I_L^{(\text{flat})}$ is the Lorentzian Einstein-Hilbert action for flat spacetime which reduces to only the surface term (the second term in Eq.(2.9)) due to $R = 0$ and $\Lambda = 0$ for flat spacetime, and $K^{(\text{flat})}$ is the trace of second fundamental form of $\partial\mathcal{M}$ in flat spacetime [13, 7]. Here note that, the integral element $\sqrt{\det h}$ in $I_L^{(\text{flat})}$ should be given by that of SdS spacetime when $I_L^{(\text{flat})}$ is used as the integration constant of action integral of SdS spacetime, because the background spacetime on which the integral in $I_L^{(\text{flat})}$ is calculated is SdS spacetime.

For our thermal equilibrium system for BEH in SdS spacetime, the region \mathcal{M} in I_L is D_b , $r_b < r < r_w$, and its boundary ∂D_b is at r_w . There is another boundary at r_b in the Lorentzian region D_b . However we do not need to consider it, because the points at r_b in the Euclidean space do not form a boundary but are the regular points when τ has the period (5.27). Then the first fundamental form h_{ij} ($i, j = 0, 2, 3$) of ∂D_b in the static chart is

$$ds^2|_{r=r_w} = h_{ij} dx^i dx^j = -f_w dt^2 + r_w^2 d\Omega^2 , \quad (5.29)$$

where

$$f_w := f(r_w) = 1 - \frac{2M}{r_w} - H^2 r_w^2 . \quad (5.30)$$

Here, since D_b is the region enclosed by BEH and heat wall, the direction of unit normal vector n^μ to ∂D_b is pointing towards CEH, $n^\mu \propto \partial_r$. Then the second fundamental form of ∂D_b in the static chart is

$$K_{ij} = \sqrt{f_w} \text{diag.} \left[-\frac{M}{r_w^2} + H^2 r_w , r_w , r_w \sin^2 \theta \right] , \quad (5.31)$$

where diag. means the diagonal matrix form.

On the other hand, the second fundamental form $K_{ij}^{(\text{flat})}$ of a spherically symmetric timelike hypersurface of radius r_w in flat spacetime is given by setting $M = 0$ and $H = 0$ in Eq.(5.31),

$$K_{ij}^{(\text{flat})} = \text{diag.} [0, r_w, r_w \sin^2 \theta] . \quad (5.32)$$

This gives $K^{(\text{flat})} := h^{ij} K_{ij}^{(\text{flat})} = 2r_w^{-1}$.

From the above, applying the Wick rotation $t \rightarrow -i\tau$ to the Lorentzian action I_L in Eq.(2.9), we obtain the Euclidean action I_{Eb} of the thermal equilibrium system for BEH via Eq.(2.7),

$$\begin{aligned} I_{Eb} &= \frac{3H^2}{8\pi} \int_{D_{Eb}} dx_E^4 \sqrt{g_E} + \frac{1}{8\pi} \int_{\partial D_{Eb}} dx_E^3 \sqrt{h_E} \left(K_E - K_E^{(\text{flat})} \right) \\ &= \frac{\beta_b}{2} [3M - r_b + 2r_w (f_w - \sqrt{f_w})] , \end{aligned} \quad (5.33)$$

where the relation for SdS spacetime $R = 4\Lambda = 12H^2$ is used in the first equality, the relation $M - H^2 r_b^3 = 3M - r_b$ due to $f(r_b) = 0$ is used in the second equality, Q_E is the quantity Q evaluated on Euclidean space, and D_{Eb} is the Euclidean region denoted by $0 \leq \tau < \beta_b$, $r_b \leq r \leq r_w$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. This I_{Eb} corresponds to $I_E[g_{Ecl}]$ in Eq.(2.16), which yields the partition function of our thermal equilibrium system for BEH.

Note that I_{Eb} should reproduce the Euclidean action (3.1) of Schwarzschild canonical ensemble as required in the working hypothesis SdS-2. To check if this is satisfied, take the limit $\Lambda \rightarrow 0$,

$$\lim_{\Lambda \rightarrow 0} I_{Eb} = 4\pi M \left[M - 2r_w (f_w - \sqrt{f_w}) \right]_{\Lambda=0} . \quad (5.34)$$

This coincides with Eq.(3.1).

5.3.2 Euclidean action for CEH

Euclidean space of thermal equilibrium system for CEH is obtained by the Wick rotations $t \rightarrow -i\tau$ in the static chart and $\eta_c \rightarrow -i\omega_c$ in the semi-global cosmological chart. These Wick rotations are equivalent, because the coordinate transformation (5.12), $\eta_c = e^{-\kappa_c r^*} \sinh(\kappa_c t)$, implies that the imaginary time ω_c in the semi-global chart is defined by $\omega_c := e^{-\kappa_c r^*} \sin(\kappa_c \tau)$, where τ is the imaginary time in the static chart. Euclidean metric in the static chart is given by Eq.(5.21). Euclidean metric in the semi-global cosmological chart is $ds_E^2 = \Upsilon_c(r) [d\omega_c^2 + d\chi_c^2] + r^2 d\Omega^2$, where Υ_c is defined in Eq.(5.14). About the semi-global chart, we get from the coordinate transformation (5.12),

$$\omega_c^2 + \chi_c^2 = \left(\frac{r}{r_b} - 1 \right)^{-\kappa_c/\kappa_b} \left(1 - \frac{r}{r_c} \right) \left(\frac{r}{r_b + r_c} + 1 \right)^{-1+\kappa_c/\kappa_b} . \quad (5.35)$$

Then, because the thermal equilibrium system for CEH is the region D_c , $r_w < r < r_c$, in Lorentzian SdS spacetime, we find the topology of Euclidean space of thermal equilibrium system for CEH is $D^2 \times S^2$. Because $\omega_c^2 + \chi_c^2 = 0$ at $r = r_c$, the center of D^2 -part is at the CEH $r = r_c$, and the boundary of D^2 -part is at the heat wall $r = r_w$. The topology of heat wall boundary is $S^1 \times S^2$, where S^1 is along the τ -direction.

Because the Lorentzian SdS spacetime is regular at r_c , the Euclidean space is also regular at r_c . Using the static chart (5.21) and defining a coordinate y_c and a function $\gamma(y_c)$ by $y_c^2 := r_c - r$ and $\gamma(y_c) := \sqrt{f(r_c - y_c^2)}$, we obtain the Euclidean metric near CEH,

$$ds_E^2 \simeq \frac{2}{\kappa_c} [y_c^2 d(\kappa_c \tau)^2 + dy_c^2] + r^2 d\Omega^2 . \quad (5.36)$$

It is obvious that the Euclidean space is regular at CEH if the imaginary time has the period β_c defined by

$$0 \leq \tau < \beta_c := \frac{2\pi}{\kappa_c} . \quad (5.37)$$

Throughout our discussion, τ has the period β_c in the Euclidean space of thermal equilibrium system for CEH.

Now we proceed to the calculation of Euclidean action I_{Ec} of the thermal equilibrium system for CEH. Lorentzian action is defined in Eq.(2.9) for the spacetime region D_c , $r_w < r < r_c$. The calculation of Euclidean action I_{Ec} is parallel to that of I_{Eb} except for the direction of unit normal vector n^μ to ∂D_c and the integration constant I_{sub} in I_L . Concerning the vector n^μ , since D_c is the region enclosed by CEH and heat wall, the direction of n^μ is pointing towards BEH, $n^\mu \propto -\partial_r$.

Concerning the integration constant, following the working hypothesis SdS-2, we determine the integration constant I_{sub} for CEH with referring to the de Sitter canonical ensemble [1] formulated in Sec.4. At the limit $M \rightarrow 0$, the term I_{sub} should reduce to the integration constant $I_{\text{sub}}^{(\text{dS})}$ of the de Sitter canonical ensemble which is read from Eqs.(4.2), (4.29) and (4.30),

$$I_{\text{sub}}(M=0) = I_{\text{sub}}^{(\text{dS})} := \left(\frac{1}{H r_w} - 1 \right) \sqrt{f_w(M=0)} I_L^{(\text{flat})}, \quad (5.38)$$

where $I_L^{(\text{flat})}$ is the action of flat spacetime used in Eq.(5.28). Note that the CEH radius in de Sitter spacetime is H^{-1} , and the factor $H r_w$ is the ratio of heat wall radius r_w to CEH radius. Hence we set I_{sub} for the CEH in SdS spacetime,

$$I_{\text{sub}} := \left(\frac{r_c}{r_w} - 1 \right) \sqrt{f_w} I_L^{(\text{flat})}, \quad (5.39)$$

where it should be emphasized that the signature of $K^{(\text{flat})}$ in the integrand of $I_L^{(\text{flat})}$ (shown in Eq.(5.28)) should be reversed, because the direction of normal vector n^μ is reversed as mentioned in previous paragraph.

Then we obtain the Euclidean action I_{Ec} of our thermal equilibrium system for CEH via Eqs.(2.9) and (2.7),

$$I_{Ec} = -\frac{\beta_c}{2} [3M - r_c + 2r_c f_w], \quad (5.40)$$

where the relation $M - H^2 r_c^3 = 3M - r_c$ due to $f(r_c) = 0$ is used, and the overall minus signature comes from the direction of normal vector n^μ to ∂D_c . This I_{Ec} corresponds to $I_E[g_{Ecl}]$ in Eq.(2.16), which yields the partition function of our thermal equilibrium system for CEH.

Note that I_{Ec} should reproduce the Euclidean action (4.31) of de Sitter canonical ensemble as required in the working hypothesis SdS-2. To check if this is satisfied, take the limit $M \rightarrow 0$,

$$\lim_{M \rightarrow 0} I_{Ec} = -\frac{\pi}{H^2} [1 - 2(H r_w)^2]. \quad (5.41)$$

This coincides with Eq.(4.31).

5.4 Breakdown of Entropy-Area Law for the Black Hole Event Horizon

We examine whether the entropy-area law holds for “consistent” thermodynamics of our thermal equilibrium system for BEH.

5.4.1 Temperature and free energy of BEH

By the assumption SdS-3, the temperature T_b of BEH is defined by Eq.(2.11), which relates T_b to the proper length in the imaginary time direction at the boundary ∂D_b (the direction along S^1 part of boundary topology $S^1 \times S^2$ in Euclidean space),

$$T_b := \left[\int_0^{\beta_b} \sqrt{f_w} d\tau \right]^{-1} = \frac{\kappa_b}{2\pi \sqrt{f_w}}, \quad (5.42)$$

where β_b is the imaginary time period (5.27) and f_w is in Eq.(5.30). Under the length size scaling (5.18), this temperature is scaled as $T_b \rightarrow \lambda^{-1} T_b$. Therefore, by the assumption SdS-2, T_b is an intensive state variable of BEH.

Note that this T_b coincides with the Hawking temperature of BEH derived originally by Gibbons and Hawking [12], and the factor $\sqrt{f_w}$ is the so-called Tolman factor [26] which expresses the gravitational redshift affecting the Hawking radiation propagating from BEH to observer at r_w . Therefore this T_b is the temperature measured by the observer at heat wall.

By the assumption SdS-2, the extensive state variable of system size for our thermal equilibrium system is the surface area of heat wall,

$$A := 4\pi r_w^2. \quad (5.43)$$

By the assumption SdS-3, the free energy F_b of BEH in Eq.(5.16) is defined by Eq.(2.16),

$$F_b(T_b, A, X_b) := -T_b I_{Eb} = r_w - r_w \sqrt{f_w} - \frac{3M - r_b}{2\sqrt{f_w}}. \quad (5.44)$$

Under the length size scaling (5.18), this free energy satisfies the scaling law of thermodynamic functions, $F_b \rightarrow \lambda F_b$. As discussed at Eq.(5.44), F_b is regarded as a function of three independent state variables T_b , A and the state variable X_b of CEH's gravitational effect on BEH. However, since X_b is not specified as mentioned in the assumption SdS-4, the form of F_b as a function of (T_b, A, X_b) remains unknown. Instead, Eq.(5.44) shows F_b as a function of independent parameters (M, H, r_w) .

Let us verify that the free energy F_b is a function of three independent state variables. We use the reductive absurdity: Assume that only two (not three) state variables are independent. This assumption means that, as for the ordinary non-magnetized gases, F_b is a function of T_b and A , $F_b(T_b, A)$. Here it is obvious from Eq.(5.42) that T_b depends on three parameters (M, H, r_w) , while A depends only on r_w . Then, via Eq.(2.21), a mathematical relation $(\partial_M F_b)/(\partial_M T_b) = (\partial_H F_b)/(\partial_H T_b)$ must hold if F_b is a function of (T_b, A) . However we find this relation does not hold, $(\partial_M F_b)/(\partial_M T_b) \neq (\partial_H F_b)/(\partial_H T_b)$, via Eq.(5.46) shown below. Hence the assumption of two independent state variables is denied by the reductive absurdity. Now the working hypothesis SdS-1, which assumes F_b to be a function of three independent state variables, is verified.

To support the discussion in previous paragraph and for later use, we show some differentials:

$$\frac{\partial T_b}{\partial M} = \frac{1}{2\pi r_b^2 f_w^{3/2}} \left[\frac{r_b}{2r_w} (1 - 3H^2 r_b^2) - \frac{1 + 3H^2 r_b^2}{1 - 3H^2 r_b^2} f_w \right], \quad (5.45a)$$

$$\frac{\partial T_b}{\partial H} = \frac{1}{4\pi H r_b f_w^{3/2}} \left[\left(1 - \frac{2M}{r_w}\right) (1 - 3H^2 r_b^2) - \frac{1 + 3H^2 r_b^2}{1 - 3H^2 r_b^2} \frac{2M f_w}{r_b} \right], \quad (5.45b)$$

and

$$\frac{\partial F_b}{\partial M} = \frac{1}{2f_w^{3/2}} \left[-\frac{3M - r_b}{r_w} + \frac{1 + 3H^2 r_b^2}{1 - 3H^2 r_b^2} f_w \right], \quad (5.45c)$$

$$\frac{\partial F_b}{\partial H} = \frac{1}{2f_w^{3/2}} \left[(2r_w + r_b - 7M - 2H^2 r_w^3) H r_w^2 + \frac{2H r_b^3 f_w}{1 - 3H^2 r_b^2} \right], \quad (5.45d)$$

where the differentials in Eqs.(5.7a) and (5.7c) are used. Then we get

$$\frac{\partial F_b}{\partial M} = -\pi r_b^2 \frac{\partial T_b}{\partial M}, \quad (5.46a)$$

$$\frac{\partial F_b}{\partial H} \not\propto \pi r_b^2 \frac{\partial T_b}{\partial H}, \quad (5.46b)$$

where the definition of r_b , $f(r_b) = 0$, is used in the first relation. These relations are used in examining the entropy-area law in next subsection.

5.4.2 Entropy of BEH

By the assumption SdS-3, the entropy S_b of BEH is defined as the thermodynamic conjugate variable to T_b ,

$$S_b := -\frac{\partial F_b(T_b, A, X_b)}{\partial T_b}. \quad (5.47)$$

Under the length size scaling (5.18), this entropy satisfies the extensive scaling law, $S_b \rightarrow \lambda^2 S_b$. By Eq.(2.19), S_b is rearranged to

$$S_b = -\frac{(\partial_M F_b)(\partial_H X_b) - (\partial_H F_b)(\partial_M X_b)}{(\partial_M T_b)(\partial_H X_b) - (\partial_H T_b)(\partial_M X_b)}. \quad (5.48)$$

Then we get by Eq.(5.46a),

$$S_b = \frac{\pi r_b^2 (\partial_M T_b)(\partial_H X_b) + (\partial_H F_b)(\partial_M X_b)}{(\partial_M T_b)(\partial_H X_b) - (\partial_H T_b)(\partial_M X_b)}. \quad (5.49)$$

This denotes the following: If $\partial_M X_b \equiv 0$, then the entropy-area law $S_b \equiv \pi r_b^2$ holds. However, if $\partial_M X_b \neq 0$, then Eq.(5.46b) together with Eq.(5.49) imply that the entropy-area law breaks down $S_b \neq \pi r_b^2$. Therefore we find that the entropy-area law holds if and only if $\partial_M X_b \equiv 0$.

In summary, although the validity of entropy-area law for BEH can not be judged at present, we can clarify the issue on the entropy-area law for BEH: *The necessary and sufficient condition to ensure the entropy-area law for BEH is that the BEH is in thermal equilibrium and the state variable X_b satisfies $\partial_M X_b \equiv 0$. If the CEH's gravitational effect on BEH, X_b , is characterized by the CEH's quantity $\kappa_c r_w$ as shown in Eq.(5.19), then $\partial_M X_b \neq 0$ and the entropy-area law breaks down for BEH. If the CEH's gravitational effect X_b is characterized by $H r_w$ as shown in Eq.(5.20), then $\partial_M X_b \equiv 0$ and the entropy-area law holds. The validity of entropy-area law for BEH will be judged by revealing which of κ_c or H is appropriate as the characteristic quantity of CEH's gravity.*

At the end of this subsection, some physical discussions which may support the breakdown of entropy-area law for BEH are given. And in next subsection, it is revealed mathematically that the entropy-area law breaks down for CEH.

5.4.3 Thermodynamic consistency of BEH

Let us confirm the ‘‘thermodynamic consistency’’ of our thermodynamics of BEH under the minimal set of assumptions. Since the concrete form of X_b is not specified, the following discussions and calculations are very formal. However we can imply that the thermodynamic consistency is satisfied. Moreover, it is also verified that X_b is an extensive variable and proportional to r_w^2 .

Following the assumption SdS-3, the internal energy E_b of BEH can be defined by the argument of statistical mechanics,

$$E_b := -\left. \frac{\partial \ln Z_{cl}}{\partial(1/T_b)} \right|_{A, X_b = \text{const.}} = \frac{\partial(F_b/T_b)}{\partial(1/T_b)} = F_b + T_b S_b, \quad (5.50)$$

where Eq.(2.16) is used in the second equality, and the definition of S_b in Eq.(5.47) is used in the third equality. Under the length size scaling (5.18), this E_b satisfies the scaling law of thermodynamic functions, $E_b \rightarrow \lambda E_b$. The third equality in Eq.(5.50), $E_b = F_b + T_b S_b$, is regarded as the Legendre transformation between F_b and E_b , which determines E_b to be a function of (S_b, A, X_b) . On the other hand, in thermodynamic argument, the internal energy is the thermodynamic function which is regarded as a function of only extensive state variables. Hence, it is verified that X_b must be an extensive state variable. (The proportionality of X_b to r_w^2 will also be shown mathematically at the end of this subsection.)

Next, in order to see the first law, we need the intensive state variables which are thermodynamically conjugate to A and X_b . By the assumption SdS-3, the conjugate state variable is defined by the appropriate differential of a thermodynamic function. In an analogy with the ordinary pressure of ordinary gases, the

surface pressure (at the heat wall) σ_b is defined formally as

$$\sigma_b := -\frac{\partial F_b(T_b, A, X_b)}{\partial A} \quad (5.51a)$$

$$\begin{aligned} &= -\frac{1}{8\pi r_w} \frac{1}{(\partial_M T_b)(\partial_H X_b) - (\partial_M X_b)(\partial_H T_b)} \\ &\times \left[\{(\partial_H F_b)(\partial_{r_w} T_b) - (\partial_{r_w} F_b)(\partial_H T_b)\} (\partial_M X_b) \right. \\ &\quad + \{(\partial_{r_w} F_b)(\partial_M T_b) - (\partial_M F_b)(\partial_{r_w} T_b)\} (\partial_H X_b) \\ &\quad \left. + \{(\partial_M F_b)(\partial_H T_b) - (\partial_H F_b)(\partial_M T_b)\} (\partial_{r_w} X_b) \right], \end{aligned} \quad (5.51b)$$

where Eq.(2.18) is used in the second equality. Under the length size scaling (5.18), this σ_b satisfies the intensive scaling law, $\sigma_b \rightarrow \lambda^{-1} \sigma_b$. The state variable given by the same definition with σ_b appears also in single-horizon thermodynamics [7, 8, 1] to ensure the thermodynamic consistency. (See Appendix B of paper [1] for thermodynamic meanings of A and σ_b .)

The intensive state variable Y_b conjugate to X_b is defined formally as

$$Y_b := \frac{\partial F_b(T_b, A, X_b)}{\partial X_b} = \frac{(\partial_M F_b)(\partial_H T_b) - (\partial_H F_b)(\partial_M T_b)}{(\partial_M X_b)(\partial_H T_b) - (\partial_H X_b)(\partial_M T_b)}, \quad (5.52)$$

where Eq.(2.19) is used in the second equality. Under the length size scaling (5.18), when X_b is scaled as an extensive variable $X_b \rightarrow \lambda^2 X_b$, then this Y_b satisfies the intensive scaling law, $Y_b \rightarrow \lambda^{-1} Y_b$.

Then by definitions of S_b , σ_b and Y_b , we get

$$dF_b(T_b, A, X_b) = -S_b dT_b - \sigma_b dA + Y_b dX_b. \quad (5.53)$$

The first law follows this relation via the Legendre transformation in Eq.(5.50),

$$dE_b(S_b, A, X_b) = T_b dS_b - \sigma_b dA + Y_b dX_b. \quad (5.54)$$

Concerning the internal energy, the Euler relation is interesting from the point of view of thermodynamics, because it gives a restriction on the form of state variables. By the scaling laws of extensive variable and thermodynamic function, we get

$$\lambda E_b(S_b, A, X_b) = E_b(\lambda^2 S_b, \lambda^2 A, \lambda^2 X_b). \quad (5.55a)$$

This denotes that $E_b(S_b, A, X_b)$ is the homogeneous expression of degree 1/2. Operating the differential ∂_λ on Eq.(5.55a), we get

$$\frac{1}{2} E_b(S_b, A, X_b) = T_b S_b - \sigma_b A + Y_b X_b. \quad (5.55b)$$

This relation (5.55b) is obtained from the scaling behavior (5.55a). Furthermore by the well-known *Euler's theorem on the homogeneous expression*, the scaling behavior (5.55a) is also obtained from the relation (5.55b) (which is proven by the vanishing differential $\partial_\lambda[\lambda^{-1} E_b(\lambda^2 S_b, \lambda^2 A, \lambda^2 X_b)] = 0$). Hence Eqs.(5.55a) and (5.55b) are equivalent. As shown below, we find the Euler relation (5.55b) is consistent with the assumption SdS-4:

By the Legendre transformation in Eq.(5.50) and the Euler relation (5.55b), we get a relation, $F_b = T_b S_b - 2\sigma_b A + 2Y_b X_b$. Then substituting Eqs.(5.48), (5.51b) and (5.52) into this relation, we obtain

$$k_1 \frac{\partial X_b}{\partial M} + k_2 \frac{\partial X_b}{\partial H} + r_w k_3 \frac{\partial X_b}{\partial r_w} = 2k_3 X_b, \quad (5.56)$$

where

$$k_1 := -F_b (\partial_H T_b) - T_b (\partial_H F_b) \quad (5.57a)$$

$$-r_w [(\partial_H F_b)(\partial_{r_w} T_b) - (\partial_{r_w} F_b)(\partial_H T_b)]$$

$$k_2 := F_b (\partial_M T_b) + T_b (\partial_M F_b) \quad (5.57b)$$

$$-r_w [(\partial_{r_w} F_b)(\partial_M T_b) - (\partial_M F_b)(\partial_{r_w} T_b)]$$

$$k_3 := -(\partial_M F_b)(\partial_H T_b) + (\partial_H F_b)(\partial_M T_b). \quad (5.57c)$$

The concrete forms of k_i ($i = 1, 2, 3$) are obtained from the differentials of T_b and F_b shown in Eq.(5.45a) \sim (5.46b), and result in relations,

$$k_1 = M k_3 \quad , \quad k_2 = -H k_3 . \quad (5.58)$$

Then Eq.(5.56) reduces to

$$M \frac{\partial X_b}{\partial M} - H \frac{\partial X_b}{\partial H} + r_w \frac{\partial X_b}{\partial r_w} = 2 X_b . \quad (5.59)$$

This partial differential equation (PDE) is equivalent to the relation (5.55b) which is also equivalent to the relation (5.55a). Therefore, if a solution X_b of our PDE (5.59) exists, then the X_b satisfies the extensive scaling behavior $X_b \rightarrow \lambda^2 X_b$ under the length size scaling (5.18). Indeed, the general solution of PDE (5.59) is expressed as

$$X_b(M, H, r_w) = r_w^2 \tilde{\psi}_b(M/r_w, H r_w) , \quad (5.60)$$

where $\tilde{\psi}_b(x, y)$ is an arbitrary function of two arguments. Obviously this X_b is proportional to r_w^2 , and satisfies the extensive scaling law under the length size scaling (5.18). It is also obvious that the arbitrary functions $\Psi_b(\kappa_c r_w)$ in Eq.(5.19) and $\Psi_b(H r_w)$ in Eq(5.20) are consistent with $\tilde{\psi}_b(M/r_w, H r_w)$ in Eq.(5.60), since κ_c in Eq.(5.5) is expressed as a function of M/r_w and $H r_w$. Hence we find that the Euler relation (5.55b) is consistent with the assumption SdS-4, which implies that the internal energy E_b and also the free energy F_b are defined well in our thermodynamics of BEH. The well-defined free energy guarantees the thermodynamic consistency. Now it has been checked that the minimal set of assumptions introduced in Sec.5.2 constructs the “consistent” thermodynamics for BEH.

5.4.4 Physical expectation of the breakdown of entropy-area law for BEH

Some comments which suggest the breakdown of entropy-area law for BEH may be possible. Let us try to give two comments: For the first, recall the meaning of state variable X_b , which expresses the thermodynamic effect on BEH due to the external gravitational field produced by CEH. Furthermore it is worth pointing out that the CEH temperature T_c depends on κ_c which has dependence on (M, H) , not on H solely. Then, in the assumption SdS-4, it may be natural that the quantity $\kappa_c r_w$, not $H r_w$, is the characteristic variable of CEH’s gravity and the extensive variable X_b is expressed by $X_b = r_w^2 \Psi_b(\kappa_c r_w)$ as required in Eq.(5.19). If this is true, then the breakdown of entropy-area law for BEH is concluded as explained at Eq.(5.49).

Next, recall that, in Sec.5.2, our thermal equilibrium systems for BEH and CEH are compared qualitatively with the magnetized gas. By the differential of free energy $F_{\text{gas}}(T_{\text{gas}}, V_{\text{gas}}, \vec{H}_{\text{ex}})$, the entropy S_{gas} and pressure P_{gas} of the gas are defined by (see for example §52, 59 and 60 in Landau and Lifshitz [23]),

$$S_{\text{gas}} := - \frac{\partial F_{\text{gas}}(T_{\text{gas}}, V_{\text{gas}}, \vec{H}_{\text{ex}})}{\partial T_{\text{gas}}} \quad , \quad P_{\text{gas}} := - \frac{\partial F_{\text{gas}}(T_{\text{gas}}, V_{\text{gas}}, \vec{H}_{\text{ex}})}{\partial V_{\text{gas}}} . \quad (5.61)$$

This implies that all state variables of the gas depend on the external field \vec{H}_{ex} . Therefore, the entropy of the gas under the influence of external magnetic field deviates from the entropy without external magnetic field. Hence, for our thermal equilibrium system for BEH, it may be naturally expected that BEH’s entropy under the influence of external gravitational field of CEH does not satisfy the entropy-area law which holds for BEH in single-horizon spacetimes.

The above two comments have no rigorous mathematical verification, but seem to be physically reasonable. Furthermore, one additional comment which support the breakdown of entropy-area law for BEH will be given in Sec.6.

5.5 Breakdown of Entropy-Area Law for the Cosmological Event Horizon

We examine whether the entropy-area law holds for “consistent” thermodynamics of our thermal equilibrium system for CEH. Discussion in this section goes parallel to Sec.5.4. However the integration constant in Euclidean action, which is determined with referring to de Sitter canonical ensemble, enables us to find a reasonable evidence of the breakdown of entropy-area law for CEH.

5.5.1 Temperature and free energy of CEH

By the assumption SdS-3, the temperature T_c of CEH is defined by Eq.(2.11), which relates T_c to the proper length in the imaginary time direction at the boundary ∂D_c ,

$$T_c := \left[\int_0^{\beta_c} \sqrt{f_w} d\tau \right]^{-1} = \frac{\kappa_c}{2\pi\sqrt{f_w}}, \quad (5.62)$$

where β_c is the imaginary time period (5.37) and f_w is in Eq.(5.30). This T_c coincides with the Hawking temperature obtained by Gibbons and Hawking [12], and the factor $\sqrt{f_w}$ is the Tolman factor [26] which expresses the gravitational redshift affecting the Hawking radiation propagating from CEH to observer at r_w . Under the length size scaling (5.18), this temperature satisfies the extensive scaling law, $T_c \rightarrow \lambda^{-1} T_c$.

As defined in assumption SdS-2, the extensive state variable of system size for our thermal equilibrium system is the surface area of heat wall,

$$A := 4\pi r_w^2. \quad (5.63)$$

By the assumption SdS-3, the free energy F_c of CEH in Eq.(5.17) is defined by Eq.(2.16),

$$F_c(T_c, A, X_c) := -T_c I_{Ec} = r_c \sqrt{f_w} + \frac{3M - r_c}{2\sqrt{f_w}}. \quad (5.64)$$

Under the length size scaling (5.18), this free energy satisfies the scaling law of thermodynamic functions, $F_c \rightarrow \lambda F_c$. Furthermore, by the same discussion given in Sec.5.4, it is also mathematically verified that the free energy F_c must be a function of three independent variables, which verifies the working hypothesis SdS-1.

Let us show some differentials for later use:

$$\frac{\partial T_c}{\partial M} = \frac{1}{2\pi r_c^2 f_w^{3/2}} \left[\frac{r_c}{2r_w} (3H^2 r_c^2 - 1) - \frac{3H^2 r_c^2 + 1}{3H^2 r_c^2 - 1} f_w \right], \quad (5.65a)$$

$$\frac{\partial T_c}{\partial H} = \frac{1}{4\pi H r_c f_w^{3/2}} \left[\left(1 - \frac{2M}{r_w} \right) (3H^2 r_c^2 - 1) - \frac{3H^2 r_c^2 + 1}{3H^2 r_c^2 - 1} \frac{2M f_w}{r_c} \right], \quad (5.65b)$$

and

$$\frac{\partial F_c}{\partial M} = \frac{\partial [(r_c - r_w) \sqrt{f_w}]}{\partial M} + \frac{1}{2f_w^{3/2}} \left[-\frac{r_c - 3M}{r_w} + \frac{3H^2 r_c^2 + 1}{3H^2 r_c^2 - 1} f_w \right], \quad (5.65c)$$

$$\begin{aligned} \frac{\partial F_c}{\partial H} &= \frac{\partial [(r_c - r_w) \sqrt{f_w}]}{\partial H} \\ &+ \frac{1}{2f_w^{3/2}} \left[-(2r_w + r_c - 7M - 2H^2 r_w^3) H^2 r_w^2 + \frac{2H r_c^3 f_w}{3H^2 r_c^2 - 1} \right], \end{aligned} \quad (5.65d)$$

where the differentials in Eqs.(5.7b) and (5.7d) are used. Then we get

$$\frac{\partial F_c}{\partial M} = \frac{\partial [(r_c - r_w) \sqrt{f_w}]}{\partial M} - \pi r_c^2 \frac{\partial T_c}{\partial M}, \quad (5.66a)$$

$$\frac{\partial F_c}{\partial H} \not\propto \pi r_c^2 \frac{\partial T_c}{\partial H}, \quad (5.66b)$$

where the definition of r_c , $f(r_c) = 0$, is used in the first relation. These relations are important to get a reasonable evidence of the breakdown of entropy-area law for CEH in next subsection.

Here one might naively expect that Eqs.(5.65c) and (5.65d) would be obtained by replacing r_b with r_c in Eqs.(5.45c) and (5.45d), and also a relation $\partial_M F_c = -\pi r_c^2 \partial_M T_c$ would be expected. However the first terms in the right-hand sides in Eqs.(5.65c), (5.65d) and (5.66a) appear, because of the difference of integration constant in Euclidean action as seen in Eqs.(5.28) and (5.39). The integration constant of I_{Ec} can not be obtained by replacing r_b with r_c in that of I_{Eb} .

5.5.2 Entropy of CEH

By the assumption SdS-3, the entropy S_c of CEH is defined as the thermodynamic conjugate variable to T_c ,

$$S_c := -\frac{\partial F_c(T_c, A, X_c)}{\partial T_c} = -\frac{(\partial_M F_c)(\partial_H X_c) - (\partial_H F_c)(\partial_M X_c)}{(\partial_M T_c)(\partial_H X_c) - (\partial_H T_c)(\partial_M X_c)}, \quad (5.67)$$

where Eq.(2.19) is used in the second equality. Under the length size scaling (5.18), this entropy satisfies the extensive scaling law, $S_c \rightarrow \lambda^2 S_c$. From this definition and the assumption SdS-4 together with Eqs.(5.66), we obtain an evidence of the breakdown of entropy-area law for CEH by the reductive absurdity as follows:

Assume that the entropy-area law holds for CEH, $S_c = \pi r_c^2$. Then Eq.(5.67) and $S_c = \pi r_c^2$ reduce to a PDE of X_c ,

$$J_M \frac{\partial X_c}{\partial M} + J_H \frac{\partial X_c}{\partial H} = 0, \quad (5.68a)$$

where Eq.(5.66a) is used, and

$$J_M := \frac{\partial F_c}{\partial H} + \pi r_c^2 \frac{\partial T_c}{\partial H} \quad , \quad J_H := -\frac{\partial [(r_c - r_w) \sqrt{f_w}]}{\partial M}. \quad (5.68b)$$

We find $J_M \neq 0$ due to Eq.(5.66b), and $J_H \neq 0$ due to $\partial_M r_c \neq 0$ and $\partial_M f_w = -2/r_w$.

For the case of $X_c = r_w^2 \Psi_c(\kappa_b r_w)$ given in Eq.(5.19) of assumption SdS-4, the PDE (5.68a) results in a contradiction as follows: Note that, because the surface gravities κ_b and κ_c are independent as functions of two variables (M, H) due to non-zero *Wronskian* $(\partial_M \kappa_b)(\partial_H \kappa_c) - (\partial_H \kappa_b)(\partial_M \kappa_c) \neq 0$, the three quantities $(\kappa_b, \kappa_c, r_w)$ can be regarded as independent variables instead of (M, H, r_w) . Here, the transformation of independent variables between two pairs (M, H, r_w) and $(\kappa_b, \kappa_c, r_w)$ is interpreted as the coordinate transformation in the state space of thermal equilibrium states of CEH. Then we find $\partial_{\kappa_c} X_c \equiv 0$ for $X_c = r_w^2 \Psi_c(\kappa_b r_w)$, and the PDE (5.68a) reduces to

$$\left(J_M \frac{\partial \kappa_b}{\partial M} + J_H \frac{\partial \kappa_b}{\partial H} \right) \frac{\partial X_c}{\partial \kappa_b} = 0, \quad (5.69)$$

which gives $\partial_{\kappa_b} X_c \equiv 0$ due to $J_M \partial_M \kappa_b + J_H \partial_H \kappa_b \neq 0$. On the other hand, the form of $X_c = r_w^2 \Psi_c(\kappa_b r_w)$ means $\partial_{\kappa_b} X_c \neq 0$, since Ψ_c is not constant as explained in assumption SdS-4. Hence we find the PDE (5.68a), which is equivalent to the entropy-area law, contradicts Eq.(5.19) of assumption SdS-4.

Next, for the case of $X_c = r_w^2 \Psi_c(M/r_w)$ given in Eq.(5.20), the PDE (5.68a) results in a contradiction as follows: With regarding the three quantities (M, H, r_w) as independent variables, Eq.(5.20) means $\partial_H X_c \equiv 0$ and the PDE (5.68a) gives $\partial_M X_c \equiv 0$ due to $J_M \neq 0$. On the other hand, the form of $X_c = r_w^2 \Psi_c(M/r_w)$ means $\partial_M X_c \neq 0$, since Ψ_c is not constant. Hence we find the PDE (5.68a), which is equivalent to the entropy-area law, contradicts Eq.(5.20) of assumption SdS-4.

The above discussions imply the breakdown of entropy-area law by the reductive absurdity under the minimal set of assumptions introduced in Sec.5.2. Now it is concluded that we find a “reasonable” evidence of the breakdown of entropy-area law for CEH in SdS spacetime, where the “reasonableness” means that our discussion retains the “thermodynamic consistency” as shown in next subsection.

5.5.3 Thermodynamic consistency of CEH

The remaining part of this subsection is for the “thermodynamic consistency” of our thermodynamics of CEH under the minimal set of assumptions. It is also verified that X_c is an extensive variable and proportional to r_w^2 . The discussion for the confirmation of thermodynamic consistency of BEH given in Sec.5.4 is applied to CEH.

By the assumption SdS-3, the internal energy $E_c(S_c, A, X_c)$ of CEH, the surface pressure σ_c at heat wall and the intensive variable Y_c conjugate to X_c are defined by

$$E_c := - \left. \frac{\partial \ln Z_{cl}}{\partial (1/T_c)} \right|_{A, X_c = \text{const.}} = \frac{\partial (F_c/T_c)}{\partial (1/T_c)} = F_c + T_c S_c, \quad (5.70)$$

$$\begin{aligned} \sigma_c &:= - \frac{\partial F_b(T_c, A, X_c)}{\partial A} \\ &= \text{Eq.(5.51b) with replacing } (F_b, X_b) \text{ with } (F_c, X_c) \end{aligned} \quad (5.71)$$

$$\begin{aligned} Y_c &:= \frac{\partial F_c(T_c, A, X_c)}{\partial X_c} \\ &= \text{Eq.(5.52) with replacing } (F_b, X_b) \text{ with } (F_c, X_c), \end{aligned} \quad (5.72)$$

where the relation in Eq.(5.70), $E_c(S_c, A, Y_c) = F_b(T_c, A, Y_c) + T_c S_c$, is regarded as the Legendre transformation, and Eqs.(2.18) and (2.19) are used in the second equalities in σ_c and Y_c . Under the length size scaling (5.18), E_c satisfies the scaling law of thermodynamic functions $E_c \rightarrow \lambda E_c$, and σ_c and Y_c satisfy the intensive scaling law $\sigma_c \rightarrow \lambda^{-1} \sigma_c$ and $Y_c \rightarrow \lambda^{-1} Y_c$. We find that, since the internal energy is a function of only extensive state variables, the state variable X_c of BEH's gravitational effect on CEH should be an extensive variable. (The proportionality of X_c to r_w^2 will also be shown mathematically at the end of this subsection.)

Then by these definitions of E_c , σ_c and Y_c together with the definition of S_c , we get the first law for CEH,

$$dE_c(S_c, A, Y_c) = T_c dS_c - \sigma_c dA + Y_c dX_c. \quad (5.73)$$

Furthermore, the scaling behavior required in assumption SdS-2 results in the same Euler relations as in Eqs.(5.55a) and (5.55b),

$$\lambda E_c(S_c, A, X_c) = E_c(\lambda^2 S_c, \lambda^2 A, \lambda^2 X_c), \quad (5.74a)$$

$$\frac{1}{2} E_b(T_c, A, X_c) = T_c S_c - \sigma_c A + Y_c X_c. \quad (5.74b)$$

These two relations are mathematically equivalent by the Euler's theorem on the homogeneous expression.

By the Legendre transformation in Eq.(5.70) and the Euler relation (5.74b), we get a relation, $F_c = T_c S_c - 2 \sigma_c A + 2 Y_c X_c$. Then substituting S_c , σ_c and Y_c into this relation, we obtain a PDE of X_c ,

$$l_1 \frac{\partial X_c}{\partial M} + l_2 \frac{\partial X_c}{\partial H} + r_w l_3 \frac{\partial X_c}{\partial r_w} = 2 l_3 X_c, \quad (5.75)$$

where l_i ($i = 1, 2, 3$) are defined formally by the same definitions of k_i in Eqs.(5.57a), (5.57b) and (5.57c) by replacing (F_b, T_b) with (F_c, T_c) . Using the differentials of T_c and F_c shown at Eq.(5.65a) \sim (5.66b), we find $l_1 = M l_3$ and $l_2 = -H l_3$ which is the same with Eq.(5.58). Then our PDE (5.75) reduces to the same PDE in Eq.(5.59), and its general solution is

$$X_c(M, H, r_w) = r_w^2 \tilde{\psi}_c(M/r_w, H r_w), \quad (5.76)$$

where $\tilde{\psi}_c(x, y)$ is an arbitrary function of two arguments. Obviously this X_c is proportional to r_w^2 , and satisfies the extensive scaling law under the length size scaling (5.18). It is also obvious that the arbitrary functions $\Psi_c(\kappa_b r_w)$ in Eq.(5.19) and $\Psi_c(M/r_w)$ in Eq.(5.20) are consistent with $\tilde{\psi}_c(M/r_w, H r_w)$ in Eq.(5.76), since κ_b in Eq.(5.5) is expressed as a function of M/r_w and $H r_w$. Hence we find that the Euler relation (5.74b) is consistent with the assumption SdS-4, which implies that the internal energy E_c and also the free energy F_c are defined well in our thermodynamics of CEH. The well-defined free energy guarantees the thermodynamic consistency. Now it has been checked that the minimal set of assumptions introduced in Sec.5.2 constructs the "consistent" thermodynamics for CEH. Hence, our conclusion that the entropy-area law breaks down for CEH is reasonable.

5.6 Supplement: Near Nariai Case

From the above, we give some reasonable evidence of the breakdown of entropy-area law for BEH and CEH in SdS spacetime. Our analysis is exact for parameter range, $0 < \sqrt{27}MH < 1$ and $r_b < r_w < r_c$, where the first inequality ensures that the BEH and CEH is non-degenerate, $r_b < r_c$. It is obvious that our discussion is true of the near Nariai case (near extremal case of SdS spacetime), $r_b \simeq r_c$ ($\Leftrightarrow \sqrt{27}MH \simeq 1$)¹³. However, note that the temperatures of horizons are equal at the exact Nariai case, $T_b = T_c$ at $r_b = r_c$. Then, finally in this section, we analyze the near Nariai case (near extremal case of SdS spacetime) of our two thermal equilibrium systems of BEH and CEH.

Note that, in the exact Nariai case, the two horizons degenerate and our two thermal equilibrium systems of horizons disappear. Hence we consider the near Nariai case as a perturbation of the exact Nariai case. We introduce two independent small parameters, δ_w and δ_{bc} , defined by

$$r_w =: 3M + \delta_w \quad , \quad r_c =: r_b + \delta_{bc} \quad , \quad (5.77)$$

where we require $\delta_{bc} \ll M$ which means the near Nariai case. The parameter δ_w controls the position of the heat wall, and satisfies, $-(3M - r_b) < \delta_w < r_c - 3M$.

In the following, we expand the temperatures and free energies of our two thermal equilibrium systems by the small parameters δ_w and δ_{bc} , in which the 0-th order values are of the Nariai limit $\delta_{bc} \rightarrow 0$ and $\delta_w \rightarrow 0$. We measure the size of the exact Nariai spacetime by the mass parameter M . Then, it is useful to introduce the following supplemental small parameters, δ_H and δ_α , defined by

$$H =: \frac{1}{\sqrt{27}M} - \delta_H \quad , \quad \alpha =: \frac{\pi}{2} - \delta_\alpha \quad , \quad (5.78)$$

where α is given by $\sin \alpha = \sqrt{27}MH$. One of three parameters (δ_{bc} , δ_H , δ_α) is independent. By the definition of α , $\sin \alpha = \sqrt{27}MH$, and Eq.(5.3), we obtain

$$2\sqrt{27}M\delta_H = \delta_\alpha^2 + O(\delta_\alpha^4) \quad , \quad \delta_{bc} = 2\sqrt{3}M\delta_\alpha \left[1 + \frac{1}{3}\delta_\alpha^2 + O(\delta_\alpha^4) \right] \quad . \quad (5.79)$$

Furthermore, from Eq.(5.30), we obtain

$$f_w = \frac{\delta_{bc}^2}{36M^2} \left[1 - 4 \left(\frac{\delta_w}{\delta_{bc}} \right)^2 \right] + O(\delta_{bc}^4) + O(\delta_{bc}^2 \delta_w) \quad . \quad (5.80)$$

The requirement $f_w > 0$ means $|\delta_w/\delta_{bc}| < 1/2$.

From the above, we obtain the near Nariai value of temperatures (5.42) and (5.62),

$$T_b = T_N \left[1 + \frac{\delta_{bc}}{9M} + O(\delta_{bc}^2) \right] \quad , \quad T_c = T_N \left[1 - \frac{\delta_{bc}}{9M} + O(\delta_{bc}^2) \right] \quad , \quad (5.81a)$$

where

$$T_N = \frac{1}{6\pi M} \left[1 - 4 \left(\frac{\delta_w}{\delta_{bc}} \right)^2 \right]^{-1/2} \left[1 + O(\delta_w) + O(\delta_{bc}^2) + O(\delta_w \delta_{bc}^2) \right] \quad . \quad (5.81b)$$

This means that, in the near Nariai case, the temperatures of our two thermal equilibrium systems are equal up to the leading term. Hence, at the leading term approximation, the total system composed of two horizons is in a thermal equilibrium state in the near Nariai case. Then, as discussed above, we expect that the entropy-area law holds at the leading term approximation in the near Nariai case. To see it, let us show the free energies (5.44) and (5.64) in the near Nariai case,

$$F_b = 3M - \frac{3M}{2} \left[1 - 4 \left(\frac{\delta_w}{\delta_{bc}} \right)^2 \right]^{-1/2} + O(\delta_{bc}) + O(\delta_w) + O(\delta_{bc} \delta_w) \quad (5.82a)$$

$$F_c = F_b - 3M \quad . \quad (5.82b)$$

¹³The metric of extremal SdS spacetime was found by Nariai [30], independently of the non-extreme SdS metric by Kottler [31].

Here one may think that the difference $3M = F_b - F_c$ results in the difference between entropies of BEH and CEH. But the definition of entropy, $S := -\partial F(T, A, X)/\partial T$, is important. Up to the leading term, the system size is $A = 4\pi(3M)^2$ and M is fixed in calculating the entropy. Therefore the difference $3M$ does not mean the difference between horizon entropies. The entropy of BEH is equal to that of CEH at the leading term approximation in the near Nariai case. However, unfortunately, we can not check if the entropy-area law recovers at the leading term approximation, because the state variables X_b and X_c are not specified and the partial derivative $-\partial F/\partial A$ can not be calculated. At present, we simply expect that the entropy-area law holds at the leading term approximation in the near Nariai limit.

6 Conclusion

6.1 Necessary and Sufficient Condition for the Entropy-Area Law

The main part of this article was Sec.5. In that section, in order to research whether the thermal equilibrium is the necessary and sufficient condition to ensure the entropy-area law, we have carefully constructed *two thermal equilibrium systems* individually for BEH and CEH in SdS spacetime, and the “consistent thermodynamics” have been obtained for BEH and CEH under the minimal set of assumptions. The need of those assumptions was discussed in Sec.2.1. In the construction of the two thermal equilibrium systems, the role of cosmological constant in the consistent thermodynamics has also been pointed out in the working hypothesis SdS-1, which has also been recognized in de Sitter thermodynamics in Sec.4.6. In our analysis, Euclidean action method was used with referring to Schwarzschild and de Sitter canonical ensembles to determine the integration constants (subtraction terms). As a result, we have found a reasonable evidence for the breakdown of entropy-area law for CEH in Sec.5.5, while the validity of the law for BEH could not be judged but the key issue on BEH’s entropy has been clarified in Sec.5.4. If the breakdown of the law for BEH is verified, then it means:

- *Thermal equilibrium of individual horizon in multi-horizon spacetime is just a necessary condition of entropy-area law.*
- *The necessary and sufficient condition of entropy-area law is the thermal equilibrium of the total system composed of several horizons in which the net energy flow among horizons disappears.*

Concerning BEH, we have already suggested two physical discussions which support the breakdown of entropy-area law for BEH. Furthermore, here we suggest an additional discussion to support the breakdown: Note that, while the CEH temperature T_c (5.62) is obtained from BEH temperature T_b (5.42) by the simple replacement of (r_b, κ_b) with (r_c, κ_c) , the CEH free energy F_c (5.64) can not be obtained from BEH free energy F_b (5.44) by such a simple replacement. This “asymmetry” of F_b and F_c is due to the asymmetry of integration constant of BEH’s Euclidean action (5.28) and that of CEH’s one (5.39). Then, it is naively expected that the coefficients l_i of PDE (5.75) do not satisfy the same relation (5.58) as k_i . However, we find at Eq.(5.75) that the relation (5.58) holds for both coefficients k_i and l_i , and the same expression of general solutions of X_b and X_c are obtained as shown in Eqs.(5.60) and (5.76). This may imply that the same consistent structure of thermodynamics holds for BEH and CEH even though the forms of free energies are asymmetric. If this implication is true, then, since the entropy-area law breaks down for CEH, the law for BEH may also break down.

6.2 Special Role of Cosmological Constant

Usually the cosmological constant Λ is not regarded as variable in the framework of horizon thermodynamics. Indeed, in the micro-canonical ensemble of de Sitter horizon (see references [15, 27] and Sec.4.1), it is not necessary to regard Λ as variable. Here note that, in general thermodynamical and statistical mechanical arguments, both micro-canonical and canonical ensembles can produce the same thermodynamic formalism of the system under consideration. Hence, the existence of micro-canonical ensemble of de Sitter thermodynamics indicates the existence of canonical ensemble of de Sitter thermodynamics. Then, as shown in Sec.4.6, we can explicitly recognize in the framework of canonical ensemble of de Sitter

thermodynamics that Λ should be regarded as a working variable. Furthermore, this is also true of SdS thermodynamics as discussed in Sec.5.2. The following role of Λ is worth emphasizing;

- *The canonical ensemble of CEH in both de Sitter and SdS spacetime constructs the “generalized” thermodynamics in which Λ behaves as a working variable, and the physical process is described by the constant Λ process.*

6.3 Two Discussions

Finally in this article, we make two discussions. One of them is on the quantum statistics of underlying quantum gravity, and the other is on SdS black hole evaporation as a non-equilibrium process. These are independent of each other.

6.3.1 Discussion about quantum statistics of gravity

There are some existing discussions on quantum nature of gravity under the existence of cosmological constant, e.g. in papers by Parikh and et al [32]. Those papers seems to be interested in some holographic principle. However, let us give a discussion from different point of view, which is rather interested in a “quantum statistical” property of gravity.

The analysis in the main text of this article is based on the Euclidean action method. This is equivalent to assume that the basic principle of statistical mechanics of ordinary laboratory systems works well in calculating the partition function of the canonical ensemble for our two thermal equilibrium systems. Here let us emphasize that the basic principle of statistical mechanics of ordinary laboratory systems is the *principle of equal a priori probabilities* [22, 23]¹⁴. Hence, if the analysis in this article and the comments in previous subsection are true (to imply the breakdown of entropy-area law for BEH), then it suggests that the principle of equal a priori probabilities results in the breakdown of entropy-area law for multi-horizon spacetimes in which horizon temperatures are not equal and a net energy flow among horizons exists. In other words, if the statistics of micro-states of quantum gravity obeys the principle of equal a priori probabilities, then the entropy-area law breaks down for the multi-horizon spacetimes.

Then what will be suggested if we adopt the other point of view? Let us dare to give priority to the entropy-area law, and assume that the entropy-area law holds for the two thermal equilibrium systems constructed in the assumption 1. Under this assumption, the discussion in previous paragraph implies that the principle of equal a priori probabilities and the Euclidean action method is not suitable to the quantum statistics of gravity in the multi-horizon spacetimes. In this case, the underlying quantum gravity should be formulated to yield the special statistic property of micro-states of gravity, which comes to obey the principle of equal a priori probability in the case of single-horizon limit [7, 8, 9, 1].

6.3.2 Discussion on SdS black hole evaporation

Turn our discussion to the second one, which is completely separated from the above discussion of quantum statistics. The second discussion is on SdS black hole evaporation process: Hereafter the heat wall introduced in the assumption 1 is *removed*. Let us note the inequality $T_b > T_c$ due to Eq.(5.6), which means the existence of a net energy flow from BEH to CEH due to the exchange of Hawking radiation emitted by two horizons. This means that, as mentioned in Sec.1, the region I in SdS spacetime is in a *non-equilibrium state*, and the SdS spacetime evolves in time due to the energy flow. This time evolution is the SdS black hole evaporation process. Here note that Hawking temperature is usually much lower than the energies E_b and E_c when the horizon is not quantum but classical size [5]. Then the evolution of BEH and CEH during the SdS black hole evaporation can be described by the so-called *quasi-static process*, in which thermodynamic states of BEH and CEH at each instant of the evolution can be approximated well by thermal equilibrium states. (Thermodynamic state of BEH evolves on a path in the state space

¹⁴ When this principle is applied to the micro-canonical ensemble, the Boltzmann’s relation $S = \ln W$ is obtained, where S and W are respectively the entropy and the number of states. And when this principle is applied to the canonical ensemble, the free energy is obtained by the relation in Eq.(2.16).

of thermal equilibrium states. Also the thermodynamic state of CEH do the same, but the path in state space on which CEH evolves is different from that of BEH.) This implies that the matter field of Hawking radiation is responsible for the non-equilibrium nature of SdS spacetime. Because the matter field is in a non-equilibrium state, the total system composed of SdS spacetime and the matter field of Hawking radiation is in a non-equilibrium state, even when the horizons are individually in equilibrium states¹⁵. If we can formulate a general non-equilibrium thermodynamics for arbitrary matter fields which are enclosed by two thermal bodies of different temperatures, then a non-equilibrium SdS thermodynamics may be obtained by applying the non-equilibrium thermodynamics to matter fields of Hawking radiation.

For non-self-interacting matters, a two-temperature non-equilibrium thermodynamics has already been constructed [11]. Therefore, under the assumption that the matter field of Hawking radiation is non-self-interacting (e.g. a minimal coupling massless scalar field ϕ satisfying $\square\phi = 0$), the non-equilibrium evolution process of SdS spacetime may be described by using the non-equilibrium thermodynamics of non-self-interacting matters [11]. Indeed, the non-equilibrium thermodynamics of non-self-interacting matters has already been applied to the evaporation process of Schwarzschild black hole [29], and has revealed the detail of evaporation process as a non-equilibrium process and verified the so-called generalized second law for the evaporation process [6]. However the non-equilibrium thermodynamics of non-self-interacting matters requires to know *a priori* the equilibrium state variables of thermal bodies among which the non-equilibrium matter field is enclosed. Therefore, before applying the non-equilibrium thermodynamics [11] to SdS spacetime, we have to specify the state variables X_b and X_c .

Finally for self-interacting matters, under the condition that its non-equilibrium nature is not so strong, some non-equilibrium thermodynamics have already been constructed [10]. Therefore, under the assumption that the strength of non-equilibrium nature is not so strong (e.g. not so large temperature difference), the non-equilibrium evolution process of SdS spacetime may be described by using an appropriate non-equilibrium thermodynamics [10]. Concerning self-interacting matters, as far as the author knows, no one has been applied the theories [10] even to Schwarzschild black hole evaporation process.

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¹⁵ The matter field of Hawking radiation is neglected in the main context of this article, because its energy scale is negligible for classical size horizons [5]. However, when we proceed to the research on the non-equilibrium nature of SdS spacetime and its time evolution, it is necessary to consider the matter field of Hawking radiation which is responsible for the non-equilibrium nature of SdS evaporation process.

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